AUTOMORPHISMS WITH SLOW DYNAMICS
(WITH AN APPENDIX IN COLLABORATION WITH JUNYI XIE)

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ABSTRACT. We study the automorphisms of compact kähler manifolds having *slow dynamics*. First, we give an upper bound on the polynomial entropy by adapting Gromov's classical argument, and we study the possible values of polynomial entropy in dimension 2 and 3. Second, we classify minimal automorphisms in dimension 2. Third, we prove that every automorphism with sublinear derivative growth is an isometry ; we give a counter-example in the C^∞ context which answers negatively a question of Artigue, Carrasco-Olivera and Monteverde on polynomial entropy. In the Appendix, written in collaboration with Junyi Xie, we classify compact kähler threefolds X with a free group of automorphisms acting freely on X.

(work in progress)

1. INTRODUCTION

1.1. **Automorphisms.** Let X be a compact Kähler manifold of dimension k. By definition, holomorphic diffeomorphisms $f\colon X \to X$ are called **automorphisms**; the group $\mathsf{Aut}(X)$ of all automorphisms is a (finite dimensional) complex Lie group, with possibly infinitely many connected components; its neutral component will be denoted $\mathsf{Aut}(X)^0$; its Lie algebra is the algebra of holomorphic vector fields on X.

Our goal is to study automorphisms whose dynamical behavior is of "low complexity". The main topics will be:

- polynomial entropy;
- growth rate of the derivative $\| Df^n \|$;
- equicontinuity of (f^n) almost everywhere;
- automorphisms acting minimally, or without periodic orbit.

We focus on low dimensional manifolds X, and state a few conjectures in higher dimension.

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1.2. **Polynomial entropy.** Let (X,d) be a compact metric space and $f:X\to X$ a continuous map. The Bowen metrics, at time n for the map f, are defined by the formula

$$d_n^f(x,y) := \max_{0\le j\le n-1} d(f^j(x), f^j(y)). \tag{1.1}$$

Define the (n,ε)**-covering number** $\mathrm{Cov}_\varepsilon(n)$ as the minimal number of balls of radius ε in the metric d_n that cover X. The **topological entropy** of the map f, $\mathrm{h_{top}}(f)\in \mathbf{R}_+\cup\{+\infty\}$, is the double limit

$$\mathrm{h_{top}}(f) := \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\left(\mathrm{Cov}_\varepsilon(n)\right). \tag{1.2}$$

It measures the exponential growth rate of the number of orbits that can be distinguished at a given precision ε during a period of observation equal to n. We are interested in the understanding of "simple" maps, with a topological entropy equal to zero. In this setting, we consider the **polynomial entropy**

$$\mathrm{h_{pol}}(f) := \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{\log n}\log\left(\mathrm{Cov}_\varepsilon(n)\right). \tag{1.3}$$

This quantity must be taken in $\mathbf{R}_+\cup\{+\infty\}$, but it will be finite for most of the systems we shall consider. The polynomial entropy has already been studied in several contexts: for integrable Hamiltonian systems by Jean-Pierre Marco [32], for Brouwer homeomorphisms by Louis Hauseux and Frédéric LeRoux [18], for various geometric situations by Patrick Bernard, Clémence Labrousse, and Marco [2, 24, 25, 26, 27, 33]. A similar notion was defined by Anatole Katok and Jean-Paul Thouvenot (see [22] and [19, 21]).

Our first result, gives an upper bound on the polynomial entropy for the automorphisms of low complexity and precise this bound in small dimensions; a version of this result will also be given for birational transformations. We refer to Sections 2 and 3 for precise statements, and in particular to Theorems 2.1, 3.2 and Proposition 4.1. We also advance on the complete computation of polynomial entropy for automorphisms of surfaces (see Sections 4 and 5). For now, this computation is not yet complete: there are automorphisms of surfaces preserving a genus 1 fibration for which we don't know yet the value of $\mathrm{h_{pol}}(f)$.

1.3. **Growth of derivatives.** In the setting of $\mathcal{C}^\infty$ diffeomorphisms $g\colon M\to M$ of compact manifolds, the growth of the derivatives, i.e. the growth of the sequence $\max_{x\in M}\parallel D(g^n)_x\parallel$ can be very slow, for instance less than n^α for every $\alpha>0$ (see Remark 7.8). In Section 7, we describe such an example: this is a variation on classical ideas due to Furstenberg (see Theorem 7.5). As an application, we obtain a counter-example to a question by Artigue, Carrasco-Olivera and Monteverde concerning polynomial entropy ([1]).

Our second goal is to show that the growth of the derivative of automorphisms does not exhibit such behaviors: if the growth of the derivative is sub-linear, then the automorphism preserves a kähler metric (Theorem 8.1).

1.4. **Minimal actions.** Another way to say that the dynamics of an automorphism is not chaotic is to suppose that "all orbits look the same". One way to do that is to assume that the action is minimal: every orbit is dense (for the euclidean topology). In the smooth setting, there are diffeomorphisms of compact manifolds (and homeomorphisms of surfaces) with positive topological entropy acting minimally. We don't know whether one can construct such an example for automorphisms of compact kähler manifolds. Here, we obtain a classification of automorphisms of surfaces satisfying one of the following density properties: f having no finite orbit; all orbits of f are Zarisky dense; all orbits of f are dense in Euclidian topology (f is minimal). This classification is obtained in Section 6. Among other results, we prove that minimal automorphisms of surfaces exist only on tori. We conjecture that this is true for any dimension, the first non-trivial case being Calabi-Yau manifolds in dimension 3.

1.5. **Appendix.** Instead of looking at the action of just one automorphism, one can consider the action of a group of transformations. Since groups of automorphisms satisfy Tits alternative, it is natural to look at non-abelian free groups, and this is what we do in the appendix, based on a joint work with Junyi Xie. We classify compact kähler manifolds of dimension ≤ 3 with a free action of a non-abelian free group.

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Part I.– Polynomial entropy : upper bound for automorphisms

2. Upper bound on the polynomial entropy

Let f be an automorphism of a compact Kähler manifold X. We shall denote its action on the cohomology of X by $f^*\colon H^*(X;\mathbf{Z}) \to H^*(X;\mathbf{Z})$. This linear action preserves the Dolbeault cohomology groups $H^{p,q}(X;\mathbf{C}) \subset H^*(X;\mathbf{C})$ and f_j^* will denote the action on $H^{j,j}(X;\mathbf{C})$ or $H^{j,j}(X;\mathbf{R})$. We define the **polynomial growth rates** $s_j(f) \in \mathbf{R}_+ \cup \{+\infty\}$ by

$$s_j(f) := \lim_{n\to+\infty} \frac{\log \parallel (f^n)_j^* \parallel}{\log(n)}.$$

We shall see that $s_j(f)$ is a non-negative integer when the topological entropy of f is equal to 0, and is infinite if $\mathrm{h_{top}}(f) > 0$. Then, we set

$$s(f) = \sum_{j=0}^{\dim_{\mathbf{C}}(X)} s_j(f) = \sum_{j=1}^{\dim_{\mathbf{C}}(X)-1} s_j(f). \tag{2.1}$$

The goal of this Section is to prove the following

Theorem 2.1 (Upper bound on polynomial entropy of automorphisms)**.** *Let X be a compact Kähler manifold. If f is an automorphism of X with* $\mathrm{h_{top}}(f) = 0$*, then* $\mathrm{h_{pol}}(f)$ *is finite and is bounded from above by the following integers*

$$\dim_{\mathbf{C}}(X) + s(f), \quad \dim_{\mathbf{C}}(X)(s_1(f)+1), \quad \dim_{\mathbf{C}}(X) \times b_2(X).$$

The proof follows Gromov's original argument providing an upper bound for the topological entropy of a holomorphic endomorphism. We denote by k the complex dimension of X.

Remark 2.2. In [31], Federico Lo Bianco proved that the sequence $j \mapsto s_j(f)$ is concave. Since $s_0(f) = 0$, and $s_{k-j}(f) = s_j(f)$, this implies $\leq s_j(f) \leq \min\{j, k-j\} \times s_1(f)$. We shall see that $s_j(f) \leq b_{2j}(X) - 1$ if $\mathrm{h_{top}}(f) = 0$.

2.1. Gromov's upper bound.

Theorem 2.3 ([16], [40, 39])**.** *Let $f\colon X \to X$ be a holomorphic endomorphism of a compact Kähler manifold X. Then* $\mathrm{h_{top}}(f) = \log\lambda(f)$*, where $\lambda(f)$ is the spectral radius of the action of f^* on the cohomology $H^*(X,\mathbf{C})$.*

In fact, Yomdin proved the lower bound $\log\lambda(f) \leq \mathrm{h_{top}}(f)$ for $\mathcal{C}^\infty$ maps of compact manifolds, and Gromov obtained the upper bound $\mathrm{h_{top}}(f) \leq \log\lambda(f)$ for holomorphic transformations of compact Kähler manifolds. Gromov's proof first relates the topological entropy to the volumes of iterated graphs $\Gamma(n)$, and then bounds them by a cohomological computation; both steps make use of the Kähler assumption. Here, $\Gamma(n)$ is the image of X under the map $x \mapsto (x, f(x), \ldots, f^{n-1}(x))$. That is,

$$\Gamma(n) := \left\{ x = (x_0, x_1, \ldots, x_n) \in X^{n+1} \mid \quad x_j = f(x_{j-1}) \right\}, \tag{2.2}$$

with $n \in \mathbf{Z}_+^*$. The iterated graphs $\Gamma(n)$ are subsets of X^{n+1}, and this set X^{n+1} is endowed with the distance

$$d_n^X(x, x') := \max_{0 \leq j \leq n} d(x_j, x_j') \tag{2.3}$$

for every pair of points x and x' in X^{n+1}. Let ε be a positive real number. By definition, the (n, ε)**-capacity** $\mathrm{Cap}_\varepsilon(n)$ is the minimal number of balls of radius ε in the metric d_n^X that cover $\Gamma(n) \subset X^{n+1}$. A set S is (n, ε)-**separated** if $d_n^X(x, y) > \varepsilon$ for every pair of elements $x \neq y$ in S, and the (n, ε)-**separation constant** $\mathrm{Sep}_\varepsilon(n)$ is the maximal number of elements in such a set.

Lemma 2.4. *For all $n \geq 1$ and $\varepsilon > 0$ we have*

$$\mathrm{Cap}_\varepsilon(n) = \mathrm{Cov}_\varepsilon(n) \quad \textit{and} \quad \mathrm{Sep}_{2\varepsilon}(n) \leq \mathrm{Cap}_\varepsilon(n) \leq \mathrm{Sep}_\varepsilon(n).$$

In particular, one can replace $\mathrm{Cov}_\varepsilon(n)$ by $\mathrm{Cap}_\varepsilon(n)$ or by $\mathrm{Sep}_\varepsilon(n)$ in the definition of the topological and polynomial entropies without changing their values.

Proof. For x and x' in $\Gamma(n)$, one has $d_n^X(x, x') = d_n^f(x_0, x_0')$, and the first equality follows. The comparison between Sep and Cap holds for every metric space. Indeed, if y_1, …, y_ℓ are 2ε-separated, two of them can not be in the same ball of radius ε; this proves $\mathrm{Sep}_{2\varepsilon}(n) \leq \mathrm{Cap}_\varepsilon(n)$. And if $\{y_1, \ldots, y_\ell\}$ is a maximal set of ε-separated points, then every point x is at a distance $\leq \varepsilon$ from one of the y_j, proving $\mathrm{Cap}_\varepsilon(n) \leq \mathrm{Sep}_\varepsilon(n)$. □

Denote by $\pi_j \colon X^{n+1} \to X$ the projection on the j-th factor, for $j = 0, \ldots, n$. Now, fix a Kähler metric on X, defined by some Kähler form κ, and put the metric on X^n which is defined by the Kähler form $\kappa_n = \sum_j \pi_j^* \kappa$. This metric differs from d_n^X (as ℓ^2 norm differs from ℓ^∞ norm). Let $\mathrm{Vol}(\Gamma(n))$ be the $2k$-dimensional volume of $\Gamma(n)$ with respect to the metric κ_n.

In order to relate $\mathrm{Vol}\Gamma(n)$ to (n, ε)-capacity, consider the following definition. Let W be a submanifold of X^n of dimension $\dim_{\mathbf{C}}(W) = d$. The ε**-density** $\mathrm{Dens}_\varepsilon(W, z)$ at a point $z \in W$ is the volume of the intersection of W with a κ_n-ball $B_z(\varepsilon)$ of radius ε centered at z:

$$\mathrm{Dens}_\varepsilon(W, z) := \mathrm{Vol}_{2d}\left(W \cap B_z(\varepsilon)\right). \tag{2.4}$$

The (ε,n)-**density** $\mathrm{Dens}_\varepsilon(W)$ is defined as the infimum

$$\mathrm{Dens}_\varepsilon(W) := \inf_{z\in W} \mathrm{Dens}_\varepsilon(W,z). \tag{2.5}$$

Set $\mathrm{Dens}_\varepsilon(n) := \mathrm{Dens}_\varepsilon(\Gamma(n))$. Then, $\mathrm{Sep}_\varepsilon(n)\mathrm{Dens}_{\frac{\varepsilon}{2}}(n) \leq \mathrm{Vol}\Gamma(n)$ for any $\varepsilon > 0$ and Lemma 2.4 gives

$$\log \mathrm{Cap}_{2\varepsilon}(n) \leq \log \mathrm{Vol}\Gamma(n) - \log \mathrm{Dens}_\varepsilon(n). \tag{2.6}$$

All this is obvious. Then, Gromov makes two crucial observations. Firstly, complex submanifolds of a compact Kähler manifold are locally minimal for the Kähler metric (Federer's theorem), and this forces a lower bound for the density:

Theorem 2.5. [16] *Fix a Kähler metric* κ *on* X*, and a real number* $\varepsilon > 0$. *There exists a positive constant* $C = C(\varepsilon,\kappa)$ *that does not depend on* n *such that* $\mathrm{Dens}_\varepsilon(n) \geq C > 0$.

In particular, the term $\log \mathrm{Dens}_\varepsilon(n)$ becomes negligeable in the Equation (2.6) when divided by n (or by $\log(n)$, see below). This provides the desired bound $\mathrm{h_{top}}(f) \leq \limsup_n n^{-1} \log \mathrm{Vol}\Gamma(n)$.

Secondly, Gromov remarks that this volume growth may be estimated by looking at the action of f on the cohomology of X. This comes from the definition of $\Gamma(n)$, and from the fact that the volume of a complex submanifold may be computed homologically in Kähler manifolds: the volume of $\Gamma(n)$ is obtained by pairing its homology class with the cohomology class of κ_n^k. We reproduce this argument below to obtain Theorem 2.1.

2.2. Proof of Theorem 2.1.

2.2.1. *Entropy versus volumes.* From Theorem 2.5 and Equation (2.6), we obtain

$$\mathrm{h_{pol}}(f) \leq \limsup_{n\to\infty} \frac{\log \mathrm{Vol}(\Gamma(n))}{\log n}. \tag{2.7}$$

Note that both parts of this inequality are infinite when $\mathrm{h_{top}}(f) > 0$.

2.2.2. *Action on cohomology.*

Lemma 2.6. *Let* g *be a* C^∞*-transformation of a compact manifold* M*. If* $\mathrm{h_{top}}(g) = 0$*, then* $g^*\colon H^*(X;\mathbf{R}) \to H^*(X;\mathbf{R})$ *is virtually unipotent: there is a positive iterate* g^m *such that all eigenvalues* $\alpha \in \mathbf{C}$ *of* $(g^m)^*$ *are equal to* 1.

Proof. Since g^* preserves the integral cohomology $H^*(M;\mathbf{Z})$ its characteristic polynomial $\chi_{g^*}(t)$ is an element of $\mathbf{Z}[t]$ with leading coefficient equal to 1. Hence, the eigenvalues of g^* are algebraic integers. Yomdin's lower bound $\log\lambda(g) \leq \mathrm{h_{top}}(g) = 0$ shows that all roots of $\chi_{g^*}(t)$ have modulus ≤ 1, and by Kronecker lemma they are roots of 1 (see [23]). Then, if m is divisible by the least common multiple of all the orders of the eigenvalues, all eigenvalues of $(g^m)^*$ are equal to 1. □

Fix a norm $\parallel\cdot\parallel$ on the cohomology groups of X, and assume that f^* is unipotent. One can find a basis of $H^{j,j}(X;\mathbf{C})$ in which f_j^* is a diagonal of Jordan blocks: the number $s_j(f)$ is the polynomial growth rate of $\parallel (f^n)_j^* \parallel$ and $s_j(f)+1$ is therefore equal to the size of the largest Jordan block of f_j^*. In particular, $s_j(f) \leq h^{j,j}(X) - 1 \leq b_{2j}(X) - 1$.

2.2.3. *Volumes of iterated graphs.* In this third step we use one more time that X is a Kähler manifold of dimension k.

Theorem 2.7 (Wirtinger)**.** *Let Y be a compact Kähler manifold with a fixed Kähler form κ_Y. If W is a complex analytic submanifold of Y of dimension d, its volume with respect to the Kähler metric is equal to*

$$\mathrm{Vol}(W) = \int_W (\kappa_Y)^d = [W]\cdot[\kappa_Y]^d.$$

With $W = \Gamma(n) \subset X^{n+1}$, we obtain

$$\mathrm{Vol}(\Gamma(n)) = \int_{\Gamma(n)} \kappa_n^k = \int_X \left(\sum_j \pi_j^*\kappa\right)^k = \int_X \left(\sum_{j=0}^n (f^j)^*\kappa\right)^k. \tag{2.8}$$

Now, let us apply the results of the previous paragraph. Since $\mathrm{h_{top}}(f^m) = m\mathrm{h_{top}}(f) = 0$ and $\mathrm{h_{pol}}(f^m) = \mathrm{h_{pol}}(f)$, we replace f by a positive iterate f^m to assume that f^* is unipotent. Then $\parallel (f^n)_1^*[\kappa] \parallel \leq C_1 \parallel [\kappa] \parallel n^{s_1(f)}$ for some uniform constant $C_1 > 0$. As a consequence, the norm of the class $[\sum_{j=0}^n (f^j)^*\kappa]$ is no more than $C' \parallel \kappa \parallel n^{s_1(f)+1}$, for some $C' > 0$, and since the cup product is a continuous, multi-linear map, we get $\mathrm{Vol}(\Gamma(n)) \leq C'' n^{k(s_1(f)+1)}$ for some $C'' > 0$. This proves two of the upper bounds of Theorem 2.1.

Lemma 2.8. *Let ℓ be an integer with $0 \leq \ell \leq k$. Then*

$$\parallel (f^{n_1})^*[\kappa] \wedge \ldots \wedge (f^{n_\ell})^*[\kappa] \parallel \leq C \parallel [\kappa] \parallel^\ell \prod_{j=1}^{\ell} (n_j - n_{j+1} - \ldots - n_\ell)^{s_j(f)}$$

for some constant $C > 0$ and every sequence of integers $n_1 \geq n_2 \geq \ldots \geq n_k \geq 0$.

Proof. First, for every class ω in $H^{j,j}(X;\mathbf{C})$, $\| (f^n)^*_j\omega \| \leq C_j n^{s_j(f)} \| \omega \|$ because the norm of the operators $(f^n)^*_j$ on $H^{j,j}(X;\mathbf{C})$ is bounded by $C_j n^{s_j(f)}$ for some positive constant $C_j, j = 1,\dots,k$. Then, to estimate $\| (f^{n_1})^*[\kappa] \wedge (f^{n_2})^*[\kappa] \|$ we write

$$\| (f^{n_1})^*[\kappa] \wedge (f^{n_2})^*[\kappa] \| = \| (f^{n_2})^* \left((f^{n_1-n_2})^*[\kappa] \wedge [\kappa] \right) \| \quad (2.9)$$

$$\leq C(n_2)^{s_2(f)}(n_1-n_2)^{s_1(f)} \| \kappa \|^2 . \quad (2.10)$$

Here the constant C is the product of C_1, C_2, and a constant D such that $\| \omega \wedge \kappa \| \leq D \| \kappa \| \| \omega \|$ for all classes ω in $H^{1,1}(X;\mathbf{C})$. This proves the lemma for $\ell = 2$, and this argument extends to other values of $\ell \leq k$. □

Now, by recursion on ℓ, there is a positive constant B_ℓ such that

$$\sum \| (f^{n_1})^*[\kappa] \wedge \dots \wedge (f^{n_\ell})^*[\kappa] \| \leq B_\ell \| [\kappa] \|^\ell \, n^{\ell+s_1(f)+\dots s_\ell(f)} \quad (2.11)$$

where the sum is over all ℓ-tuples (n_i) such that $n \geq n_1 \geq n_2 \geq \dots \geq n_\ell \geq 0$ With $\ell = k$ we obtain $\mathrm{Vol}(\Gamma(n)) \leq B' n^{k+s(f)}$ for some $B' > 0$. This concludes the proof of Theorem 2.1.

In the following Section, we adapt the proof of such an upper bound to the case of meromorphic transformation. Let us make a following last remark: unfortunately, one can not adapt the lower bound argument for the polynomial entropy by following the arguments of Yomdin for the proof of Theorem 2.3 since some of the terms in his bounds have exponential growth. Although, Bowen-Manning arguments can be adapted to prove the following

Lemma 2.9. *Let X be a compact manifold X, $\Gamma = \pi_1(X) =< S >$, S being a symmetric set of generators. Let f be a homeomorphism $f : X \to X$, and $\rho(f^*) := \limsup_{n\to\infty} \frac{\log \mathrm{diam}(f^*)^n(S)}{\log n}$. Then the following lower bound holds:* $\mathrm{h_{pol}} f \geq \rho(f^*) - 1$.

3. GENERALIZATION FOR MEROMORPHIC TRANSFORMATIONS

In this section, we explain how to extend Theorem 2.1 to the case of meromorphic transformations. To do it, we just have to replace Gromov's argument by a result of Dinh and Sibony, the drawback being the difficulty to estimate the growth of the volumes of the iterated graphs for meromorphic transformations.

3.1. **Growth on cohomology, graphs and entropy.** Let g be a meromorphic transformation of a compact Kähler manifold X of dimension k; let $\mathrm{Ind}(g)$ be its indeterminacy locus. Denote by $(g)^*_j$ the linear action of g on the cohomology group $H^{j,j}(X,\mathbf{C}) \subset H^*(X,\mathbf{C})$ (see [11, 17] for a definition), and fix a norm on $H^*(X,\mathbf{C})$. For every $n \geq 0$, denote by $\| (g^n)^*_j \|$ the norm of the linear transformation $(g^n)^*_j$, define $s_j(g) \in \mathbf{R}_+ \cup \{+\infty\}$ by

$$s_j(g) = \limsup_{n\to+\infty} \frac{\log \| (g^n)^*_j \|}{\log(n)} \tag{3.1}$$

and set $s(g) := s_1(g) + \ldots + s_k(g)$. Since $(g^{n+m})^*_j$ does not coincide with $(g^m)^*_j \circ (g^n)^*_j$ in general, it is not clear whether this supremum limit is actually a limit. The j**-th dynamical degree** is

$$\lambda_j(g) = \limsup_{n\to+\infty} \| (g^n)^*_j \|^{1/n},$$

and this supremum limit is actually a limit (see [11]). In these definitions of $s_j(g)$ and $\lambda_j(g)$, we could replace $\| (g^n)^*_j \|$ by $\int_X (g^n)^*(\kappa^j) \wedge \kappa^{k-j}$ for any fixed Kähler form on X; this would not change the result (see [11, 8]).

Remark 3.1. Except in dimension ≤ 2, we don't know whether $s_j(g)$ is an integer when $\lambda_j(g) = 1$; it could a priori be the case that $\| (g^n)^*_j \|$ grows like $\exp(\sqrt{n})$ or $n^{\sqrt{3}}$ as n goes to $+\infty$. We refer to [38] for this type of questions, and to [11, 8] for the main properties of $\| (g^n)^*_j \|$.

By definition, the **iterated graph** $\Gamma(n)$ of g is the closure of the set of points

$$(x, g(x), \ldots, g^n(x)) \in X^{n+1}$$

such that $x \notin \mathrm{Ind}(g)$, $g(x) \notin \mathrm{Ind}(g)$, …, $g^{n-1}(x) \notin \mathrm{Ind}(g)$. To define the notions of entropy, we need to take care of the indeterminacy locus $\mathrm{Ind}(g)$. As in [11, 17], we simply use (n,ε)-separated sets (as in § 2.1 and Lemma 2.4), but for orbits avoiding $\mathrm{Ind}(g)$; this way, we can talk of topological or polynomial entropy.

3.2. **A bound on the polynomial entropy.** The results of Theorem 2.1 can be extended to meromorphic transformations, as follows.

Theorem 3.2 (Upper bound on polynomial entropy of meromorphic transformations)**.** *Let X be a compact Kähler manifold. If $g\colon X \dashrightarrow X$ is a meromorphic transformation of X, then*

$$\mathrm{h_{pol}}(g) \leq \dim_{\mathbf{C}}(X) + \sum_{j=0}^{\dim_{\mathbf{C}}(X)} s_j(g) \leq \sum_{j=0}^{\dim_{\mathbf{C}}(X)} h^{j,j}(X).$$

Note that $s_0(g) = 0$ and for $k = \dim_{\mathbf{C}}(X)$ the number $\lambda_k(g)$ coincides with the topological degree of g; thus, if the topological degree of g is at least 2, then

$s_k(g) = +\infty$ and the theorem is empty. Consequently, we can assume from the start that g is a bimeromorphic transformation of X.

Sketch of Proof. The first steps of Gromov's argument remain valid: if one defines the polynomial growth of the iterated graphs by

$$\mathrm{povol}(g) := \limsup_{n\to+\infty} \frac{\log \mathrm{Vol}(\Gamma(n))}{\log(n)}, \tag{3.2}$$

then $\mathrm{h}_{\mathrm{pol}}(g) \le \mathrm{povol}(g)$. Our goal is to show that $\mathrm{povol}(g)$ is bounded from above by $k + s_1(g) + \ldots + s_k(g)$, where $k = \dim_{\mathbf{C}}(X)$. For this, we simply copy the argument of [12]. More precisely, replacing Lemma 2 of [12] by Corollary 1.2 of [11], the results of [12] remain valid on compact Kähler manifolds. Thus, there is a positive constant C, which depends only on the geometry of X, such that

$$\parallel f^*T \parallel \le C \parallel f_j^* \parallel \parallel T \parallel \tag{3.3}$$

for every meromorphic map $f \colon X \to X$ and every closed positive current T of bi-degree (j,j) on X. Here, $\parallel T \parallel$ is the mass of T, computed with respect to a fixed Kähler form κ: $\parallel T \parallel = \langle T | \kappa^{k-j} \rangle$. And f^*T is the positive current which is defined on $X \setminus \mathrm{Ind}(f)$ by pull-back; this upper bound on the mass of f^*T implies that the extension of f^*T by 0 on $\mathrm{Ind}(f)$ is a closed positive current, with the same mass: we shall also denote by f^*T this current. Then, as in the proof of Lemma 5 of [12], or Lemma 2.8 above, we obtain the following estimate: for every integer $0 \le \ell \le k$, and every decreasing sequence of integers $n_1 \ge n_2 \ge \ldots \ge n_k$,

$$\parallel (g^{n_1})^*[\kappa] \wedge \ldots \wedge (g^{n_\ell})^*[\kappa] \parallel \le C' \prod_{j=1}^{\ell} \parallel (g^{n_j - n_{j+1} - \ldots - n_\ell})_j^* \parallel \parallel [\kappa] \parallel \tag{3.4}$$

for some constant $C' > 0$. By definition, for every $\eta > 0$ and for m larger than some integer $m(\eta; j)$ we have $\parallel (g^m)_j^* \parallel \le m^{s_j(g)+\eta}$. By recursion on ℓ, we get

$$\parallel (g^{n_1})^*[\kappa] \wedge \ldots \wedge (g^{n_\ell})^*[\kappa] \parallel \le C'' n^{\ell(1+\eta)+s_1(g)+\ldots s_\ell(g)} \parallel [\kappa] \parallel \tag{3.5}$$

for some constant C''. To deduce the result, take $\ell = k$ and let η go to 0. $\square$

Part II.– Polynomial entropy in dimension 2

4. POLYNOMIAL ENTROPY IN SMALL DIMENSIONS

Proposition 4.1 (Small dimension automorphisms)**.** *Let X be a compact Kähler manifold of dimension* $\dim_{\mathbf{C}}(X) \leq 3$. *If $f \in \mathsf{Aut}(X)$ satisfies* $\mathrm{h}_{\mathrm{top}}(f) = 0$ *then* $\mathrm{h}_{\mathrm{pol}}(f) \leq \dim_{\mathbf{C}}(X)^2$.

The goal of this section is to prove this Proposition.

4.1. **Curves and surfaces.** If X is a curve, and f is an automorphism of X, the action of f on $H^2(X;\mathbf{C})$ is just the identity, Theorem 2.1 provides the upper bound $\mathrm{h}_{\mathrm{pol}}(f) \leq \dim_{\mathbf{C}}(X) = 1$. In fact, if the genus of X is positive, then f is an isometry (for the euclidean or hyperbolic metric), and $\mathrm{h}_{\mathrm{pol}}(f) = 0$. If the genus of X is 0, then f is given by a Möbius transformation, and either $\mathrm{h}_{\mathrm{pol}}(f) = 1$, or f is conjugate to an element of $PU_2(\mathbf{C}) \subset PGL_2(\mathbf{C})$ and then $\mathrm{h}_{\mathrm{pol}}(f) = 0$. In particular, if $\mathrm{h}_{\mathrm{pol}}(f) = 0$, then f is an isometry for some riemmanian metric.

Suppose now that f is a bimeromorphic transformation of a compact Kähler surface X. As we shall see in Section 5, either $\lambda_1(f) > 1$, or $\lambda_1(f) = 1$ and $s_1(f) \in \{0,1,2\}$. Thus, we obtain the following result: *If f is a bimeromorphic transformation of a compact Kähler surface, either $\lambda_1(f) > 1$, or $\mathrm{h}_{\mathrm{pol}}(f) \leq 4$.* Proposition 4.1 follows from this statement and Yomdin's theorem when $\dim_{\mathbf{C}}(X) = 2$.

Remark 4.2. On surfaces, one shall also prove that f is an isometry when $\mathrm{h}_{\mathrm{pol}}(f) = 0$.

4.2. **Threefolds.** Here we use the results of Federico Lo Bianco (see Section 6.2 in [30] as well as [29] and Theorem A in [31]):

Theorem 4.3 (Federico Lo Bianco)**.** *Let $f : X \to X$ be an automorphism of a compact Kähler manifold X of dimension $k = 3$. Assume that the action of f^* on $H^2(X,\mathbf{C})$ is unipotent; then, it has a unique Jordan block of maximal size ℓ_1 localized in $H^{1,1}(X,\mathbf{C})$ and $\ell_1 \leq 5$. The other Jordan blocks in $H^2(X,\mathbf{C})$ have size $\leq \frac{\ell_1+1}{2} \leq 3$.*

Since, by duality, the action of f^* on $H^{2,2}(X;\mathbf{C})$ has Jordan blocks of the same size, we obtain $s_1(f) = s_2(f) \leq 4$, and thus $\mathrm{h}_{\mathrm{pol}}(f) \leq 3 + 2 \times 4 = 11$. We want to improve this inequality to $\mathrm{h}_{\mathrm{pol}}(f) \leq 9$, and for that we use one extra ingredient from the proof of Lo Bianco's result. Namely, we can find a basis $(u_1, \ldots, u_{\ell_1})$ for the maximal Jordan block of f_1^* that satisfies

(1) $f^*u_1 = u_1$ and $f^*u_m = u_m + u_{m-1}$ for any $m = 2, \ldots, l_1$ (normal form of the Jordan block);
(2) $u_1 \wedge u_1 = u_1 \wedge u_2 = 0$ in $H^{2,2}(X;\mathbf{C})$.

From this result, we can now estimate the volume of $\Gamma(n)$:

$$\mathrm{Vol}(\Gamma(n)) = \int_X \left(\sum_{j=0}^{n} (f^j)^*\kappa \right)^3 \leq 6 \sum_{i \leq j \leq k = 0}^{n} \int_X (f^i)^*\kappa \wedge (f^j)^*\kappa \wedge (f^k)^*\kappa;$$

since the topological degree of f is 1, we can set $j = i + t_1$ and $k = i + t_2$ to obtain

$$\mathrm{Vol}(\Gamma(n)) \leq 6 \sum_{i=0}^{n} \sum_{t_1=0}^{n-i} \sum_{t_2=0}^{n-i} \int_X \kappa \wedge (f^{t_1})^*\kappa \wedge (f^{t_2})^*\kappa.$$

Denote by $\ell_1 > \ell_2 > \ldots$ the sizes of the Jordan blocks of f^* on $H^{1,1}(X;\mathbf{C})$. Then, represent the Kähler form κ as a linear combination of vectors $\kappa = \sum_{i=1}^{s_1} \alpha_i u_i + \sum_{m=1}^{M_2} \sum_{j=1}^{s_2} \beta_i^m v_i^m + \ldots$. Here M_2 is the number of Jordan blocks of size ℓ_2 and the vectors $\{v_i^m\}_{i=1}^{s_2}$ form a basis of the corresponding invariant subspaces in $H^{1,1}(X,\mathbf{C})$. Then write out the wedge product $(f^{t_1})^*\kappa \wedge (f^{t_2})^*\kappa$: since $u_1 \wedge u_1 = u_1 \wedge u_2 = 0$ and $\ell_2 \leq 3$, we see that the form $(f^{t_1})^*\kappa \wedge (f^{t_2})^*\kappa$ is a polynomial $P(t_1,t_2)$ in t_1 and t_2 with values in $H^{2,2}(X,\mathbf{C})$ and of degree at most 6. From this, we get an upper bound

$$\mathrm{Vol}\Gamma(n) \leq \sum_{i=0}^{n} \sum_{t_1=0}^{n-i} \sum_{t_2=0}^{n-i} \int_X \kappa \wedge P_6(t_1,t_2) \leq C' n^9$$

for some positive constant C'. This shows that *an automorphism of a compact Kähler manifold of dimension* 3 *has polynomial entropy* ≤ 9 *if its topological entropy vanishes.* This concludes the proof of Proposition 4.1.

4.3. **A conjecture.** We wonder whether the bound k^2 is optimal in all dimensions:

Question 4.1. Let f be an automorphism of a compact Kähler manifold X of dimension k. If $\mathrm{h_{top}}(f) = 0$ does it follow that $\mathrm{h_{pol}}(f) \leq k^2$? Is such an upper bound optimal, in every dimension k ?

The following proposition shows that this upper bound is satisfied for automorphisms of tori. We provide a cohomological proof, and then give a second, more precise statement in Proposition 4.6.

Proposition 4.4. *If X is a complex torus of dimension k, and the automorphism $f\colon X \to X$ satisfies $\mathrm{h_{top}}(f) = 0$, then $\mathrm{h_{pol}}(f) \leq k^2$.*

Proof. We fix a Kähler form κ, and we want to bound:

$$\begin{aligned} \mathrm{Vol}(\Gamma(n)) &= \int_X \left(\sum_{j=0}^{n} (f^j)^* \kappa \right)^k && (4.1) \\ &= \sum_{i=0}^{n} \sum_{t_1=0}^{n-i} \cdots \sum_{t_{k-1}=0}^{n-i} [\kappa] \wedge (f^{t_1})^*[\kappa] \wedge \ldots \wedge (f^{t_{k-1}})^*[\kappa]. && (4.2) \end{aligned}$$

The automorphism f is acting linearly on the complex torus X by a matrix A, its action on $H^{1,0}(X,\mathbf{C})$ is given by the transposed matrix A^t, and on $H^{0,1}(X,\mathbf{C})$ by the matrix $\bar{A}^t$. If $\mathrm{h}_{\mathrm{top}}(f) = 0$ then f^* is (virtually) unipotent. Fix a basis $(u_1, \ldots, u_k)$ of $H^{1,0}(X,\mathbf{C})$ in which A^t has a canonical Jordan form, and its biggest Jordan block corresponds to the subspace generated by $(u_1, \ldots, u_{\ell_1})$; in particular $\ell_1 \leq k$. Writing $[\kappa] = \sum_{m,n} \alpha_{m,n} u_m \wedge \bar{u}_n$, we obtain

$$(f^j)^*[\kappa] = \sum_{m,n=0}^{k} \alpha_{m,n} p_{m-1}(j) \bar{p}_{n-1}(j) u_m \wedge \bar{u}_n, \tag{4.3}$$

where the $\alpha_{m,n}$ are complex numbers and the $p_\delta(j)$ are polynomial functions of degree δ in the variable j. The maximal degree in the right-hand side of (4.3) is $2(\ell_1 - 1)$. Since $u_j \wedge u_j = 0$ for all $1 \leq j \leq k$, the sum in Equation (4.1) is bounded by

$$\begin{aligned} \mathrm{Vol}(\Gamma(n)) &\leq C \sum_{i=0}^{n} \sum_{t_1=0}^{n-i} \cdots \sum_{t_{k-1}=0}^{n-i} t_1^{2(\ell_1-1)} t_2^{2(\ell_1-2)} \cdots t_{k-1}^{2\cdot 1} && (4.4) \\ &= Cn^{1+\ell_1-1+\frac{2\cdot\ell_1(\ell_1-1)}{2}} = Cn^{\ell_1^2} \leq Cn^{k^2} && (4.5) \end{aligned}$$

for some $C > 0$. □

Let F be the element of $\mathsf{SL}_k(\mathbf{Z})$ given by a Jordan block of size k, which means that $F(u_1) = u_1$ and $F(u_m) = u_m + u_{m-1}$ for every $2 \leq m \leq k$ in the canonical basis (u_j) of $\mathbf{Z}^k$. This transformation induces a diffeomorphism of the torus $\mathbf{R}^k/\mathbf{Z}^k$ (resp. of the torus $(\mathbf{C}/\Lambda)^k$ for every elliptic curve $\mathbf{C}/\Lambda$).

Lemma 4.5. *The polynomial entropy of the diffeomorphism $F : \mathbf{R}^k/\mathbf{Z}^k \to \mathbf{R}^k/\mathbf{Z}^k$ (resp. of $\mathbf{C}/\Lambda$) is equal to $k(k-1)/2$ (resp. to $k(k-1)$).*

Proof of Lemma 4.5. First note that in the canonical basis (u_j) the iterated map $F^n \in SL_k(\mathbf{Z})$ is given by the $k\times k$ matrix of the form

$$\begin{pmatrix} 1 & Q_1(n) & Q_2(n) & \dots & Q_{k-1}(n) \\ 0 & 1 & Q_1(n) & \dots & Q_{k-2}(n) \\ 0 & 0 & 1 & \dots & Q_{k-3}(n) \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

where each $Q_j(n)$ is a polynomial function in the variable n such that $Q_j(n) \approx \frac{n^j}{j!}$ up to lower degree terms. For simplicity, we set $X := (\mathbf{R}/\mathbf{Z})^k$ and choose the l_∞ metric on X. Consider the following set of points $S_n \subset X$:

$$S_n := \left\{ \frac{\varepsilon}{k}\left(i_1, \frac{i_2}{Q_1(n)}, \dots, \frac{i_k}{Q_{k-1}(n)}\right) \in X \mid i_j \in \mathbf{Z}, 0 \le i_j \le \left\lfloor Q_{j-1}(n)\frac{k}{\varepsilon}\right\rfloor \right\}.$$

Then,

$$|S_n| = \prod_{j=1}^{k}\left(\left\lfloor Q_{j-1}(n)\frac{k}{\varepsilon}\right\rfloor + 1\right) \approx \left(\frac{k}{\varepsilon}\right)^k \frac{1}{\prod_{j=1}^{k-1} j!} n^{1+2+\ldots+(k-1)},$$

where the last equivalence holds true up to the terms of lower order in n. This set is $\lambda\varepsilon$-separated for any $\lambda > 1$, but not for $\lambda \in [0,1]$. The Bowen balls of radius ε centered at the points of S_n cover X and, at the same time, all the points in the set S belong to different Bowen balls of radius $\varepsilon/2$. Thus $\mathrm{Cov}_\varepsilon(n) \simeq |S_n|$ and from definition from Equation (1.3) we get $\mathrm{h}_{\mathrm{pol}}F = 1+2+\ldots+(k-1) = k(k-1)/2$. □

Proposition 4.6. *If X is a complex torus of dimension k, and $f \in \mathrm{Aut}(X)$ satisfies $\mathrm{h}_{\mathrm{top}}f = 0$, then $\mathrm{h}_{\mathrm{pol}}(f) \le k(k-1)$.*

Proof. Write $X = \mathbf{C}^k/\Lambda$ for some co-compact lattice $\Lambda \subset \mathbf{C}^k$. There is a matrix $A \in \mathsf{GL}_k(\mathbf{C})$ and a vector $B \in \mathbf{C}^k$ such that $f(z) = A(z) + B \mod \Lambda$. The Bowen distance d_n^f does not depend on B, so we assume $B = 0$ for simplicity. Since $\mathrm{h}_{\mathrm{top}}(f) = 0$, we can replace f by a positive iterate to assume that the action of f on $H^1(X;\mathbf{R})$ is unipotent.

Consider $\mathbf{C}^k$ as a real vector space $V_{\mathbf{R}}$ of dimension $2k$; fixing a basis of Λ, we identify Λ with the lattice $\mathbf{Z}^{2k} \subset V_{\mathbf{R}} \simeq \mathbf{R}^{2k}$ and denote by $V_{\mathbf{Q}} \simeq \mathbf{Q}^{2k}$ the rational subspace $\Lambda \otimes_{\mathbf{Z}} \mathbf{Q}$. Since A is a unipotent endomorphism of $V_{\mathbf{Q}}$, there is basis of $V_{\mathbf{Q}}$ in which the matrix of A is a diagonal of Jordan blocks. Since the endomorphism is induced by a $\mathbf{C}$-linear transformation, the blocks come in pairs of the same sizes, so that the list of sizes can be written $k_1 \ge k_2 \ge k_3 \dots$

with $k_{2i+1} = k_{2i+2}$ for every $i \geq 0$. Now, the proof of Lemma 4.5 and the additivity $\mathrm{h_{pol}}(g \times h) = \mathrm{h_{pol}}(g) + \mathrm{h_{pol}}(h)$ give

$$\mathrm{h_{pol}}(f) = \sum_{i \geq 0} k_{2i+1}(k_{2i+1} - 1). \tag{4.6}$$

Since $\sum_j k_j = 2k$ and $a(a-1) + b(b-1) \leq (a+b)(a+b-1)$ for positive integers, we obtain $\mathrm{h_{pol}}(f) = k(k-1)$. □

Question 4.2. Let X be a complex Kähler surface, and let $f\colon X \to X$ be an automorphism. If $\mathrm{h_{top}}(f) = 0$, does it follow that $\mathrm{h_{pol}}(f) \leq 2$? Equivalently, if f preserves a genus 1 fibration, does it follow that $\mathrm{h_{pol}}(f) = 2$?

The equivalence between the two questions follows from the results of the next section. We hope to answer this question in the final version of this draft, and thus complete the study of the spectrum of polynomial entropy for automorphisms of surfaces.

5. Automorphisms of surfaces: classification and lower bounds

Let f be an automorphism of a compact kähler surface. The main goal of this section if to prove that, if some iterate of f is in $\mathsf{Aut}(X)^0$, then $\mathrm{h_{pol}}f \in \{0,1,2\}$. To state a more precise result, we say that f has a **wandering saddle configuration** if f has a saddle fixed point x, together with two open subsets U_1 and U_2 in X such that

(a) $\mathsf{U}_1 \cap \mathsf{U}_2 = \emptyset$;
(b) $f^n(\mathsf{U}_i) \cap U_i = \emptyset$ for $i = 1,2$ and all $n \neq 0$;
(c) $\mathsf{U}_1 \cap W^s(x) \neq \emptyset$ and $\mathsf{U}_2 \cap W^u(x) \neq \emptyset$,

where $W^s(x)$ and $W^u(x)$ denote the stable and unstable manifolds of x. See Figure 5 for illustration.

Theorem 5.1. *Let X be a compact kähler surface, and let f be an automorphism of X. If some positive iterate of f is in $\mathsf{Aut}(X)^0$, then $\mathrm{h_{pol}}f \in \{0,1,2\}$. Moreover, $\mathrm{h_{pol}}(f) = 0$ if and only if f preserves a kähler metric and $\mathrm{h_{pol}}(f) = 2$ if and only if f has a wandering saddle configuration.*

After proving this result, we shall give a general statement for all automorphisms of surfaces in Section 5.5

5.1. Preliminaries.

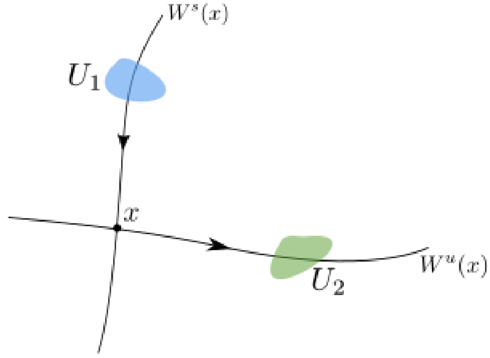

FIGURE 1. Wandering saddle connection of a saddle fixed point x.

5.1.1. *Cohomological bound.* A positive iterate of f is in $\mathsf{Aut}(X)^0$ if and only if the sequence $(\deg(f^n))$ is bounded; in that case $s_1(f) = s_2(f) = 0$ and Theorem 2.1 gives $\mathrm{h}_{\mathrm{pol}}(f) \leq 2$. The following example shows that 0, 1, and 2 are realizable. So, the main point is to prove that $\mathrm{h}_{\mathrm{pol}}(f) \in \mathbf{Z}$ when $f \in \mathsf{Aut}(X)^0$.

Example 5.2. An automorphism $g \in PGL_2(\mathbf{C})$ of $\mathbb{P}^1(\mathbf{C})$ satisfies $\mathrm{h}_{\mathrm{pol}}(g) = 1$, except when g is (conjugate to) a rotation in which case $\mathrm{h}_{\mathrm{pol}}(g) = 0$. Now, consider the group of automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$; this group contains $\mathsf{PGL}_2(\mathbf{C}) \times \mathsf{PGL}_2(\mathbf{C})$ as a subgroup of index 2. By additivity of polynomial entropy for products, we see that $\{0, 1, 2\}$ is exactly the set of possible polynomial entropies for automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, the three values 0, 1, and 2 are already realized on $\mathbb{P}^1 \times \mathbb{P}^1$.

5.1.2. *Compact groups and Kodaira dimension.* Fix a kähler metric on X, given by a kähler form κ_0. If f is contained in a compact subgroup K of $\mathsf{Aut}(X)$, and μ is a Haar measure on K, the kähler form $\kappa = \int_K g^*\kappa_0 \, d\mu(g)$ is f-invariant; in particular, f preserves a kähler metric if f has finite order.

Lemma 5.3. *Let X be a compact kähler surface. If the Kodaira-dimension of X is ≥ 0, the group $\mathsf{Aut}(X)^0$ is a compact Lie group.*

This well known result shows that $\mathsf{Aut}(X)^0$ preserves a kähler metric when $kod(X) \geq 0$. Thus, in what follows, we assume that $kod(X) = -\infty$, and we separate two cases:

- Irrational, ruled surfaces are studied in Section 5.2
- Rational surfaces are studied in Sections 5.3 and 5.4.

5.1.3. *Two lemmas.* We shall apply two basic lemmas. The first one is similar to the study of twist maps in dimension 2 by Marco [32]. We say that a function is holomorphic on the closed unit disk $\overline{\mathbb{D}}$ if it is the restriction of a holomorphic function defined on a slightly larger open disk.

Lemma 5.4. *Let $a\colon \overline{\mathbb{D}} \to \mathbf{C}^*$ be a holomorphic function that does not vanish. Let f be the transformation of $\overline{\mathbb{D}} \times \mathbb{P}^1(\mathbf{C})$ defined by $f(x,[y_0 : y_1]) = (x,[a(x)y_0 : y_1])$. Then, $\mathrm{h_{pol}}(f) \in \{0,1\}$ and $\mathrm{h_{pol}}(f) = 0$ if and only if $a(x) = c$ for some constant $c \in \mathbf{C}$ of modulus 1.*

Let $b\colon \overline{\mathbb{D}} \to \mathbf{C}^*$ be a holomorphic function. Let g be the transformation of $\overline{\mathbb{D}} \times \mathbb{P}^1(\mathbf{C})$ defined by $g(x,[y_0 : y_1]) = (x,[y_0 + b(x)y_1 : y_1])$. Then, $\mathrm{h_{pol}}(g) \in \{0,1\}$ and $\mathrm{h_{pol}}(g) = 0$ if and only if $g = \mathsf{Id}$.

Proof. The n-th iterate of g is $g^n(x,[y_0 : y_1]) = (x,[y_0 + nb(x)y_1 : y_1])$. For $p = (x,[y_0 : y_1])$ and $p' = (x',[y'_0 : y'_1])$ in $\mathbb{P}^1(\mathbf{C}) \setminus [1:0]$, one can write $Y := \frac{y_0}{y_1}$, $Y' := \frac{y'_0}{y'_1}$, and one gets

$$\mathrm{dist}(g^n(p), g^n(p')) = \max\left(\left|x - x'\right|, \left|Y - Y' + n\left(b(x) - b(x')\right)\right|\right), \qquad (5.1)$$

where the distance dist on $\overline{\mathbb{D}} \times \mathbf{C}$ is given by $\max(|x-x'|,|y-y'|)$. Moreover $|b(x) - b(x')| \leq C|x-x'|$, with $C = \max|b'(z)|$ for $z \in \mathbb{D}$. Since the distance on $\overline{\mathbb{D}} \times \mathbb{P}^1(\mathbf{C})$ is bounded from above by the distance dist on $\mathbb{D} \times \mathbf{C}$, we get the upper bound $\mathrm{h_{pol}}(g) \leq 1$. If b vanishes identically then g is the identity and $\mathrm{h_{pol}}(g) = 0$; otherwise, there are wandering points and thus $\mathrm{h_{pol}}(g) = 1$.

Now, consider the map f. If $a(x) = c$ for some constant of modulus 1, then $\mathrm{h_{pol}}(f) = 0$ since f is a product of isometries. Otherwise there is a point $x \in \mathbb{D}$ for which $|a(x)| \neq 1$, almost all points $(x,[y_0 : y_1])$ are wandering, and then $\mathrm{h_{pol}}(f) \geq 1$. For the upper bound, one remarks that there is a continous conjugacy between f and a transformation f_0 of $\overline{\mathbb{D}} \times \mathbb{P}^1(\mathbf{C})$, the derivative of which satisfies $\parallel D(f_0^n) \parallel \leq |n|$. Indeed, if h is a transformation of the sphere with a north-south dynamics, one can conjugate it to a new transformation h_0 which is infinitely tangent to the identity at each pole, and if h is close to a rotation in the $\mathcal{C}^0$-topology, so is h_0. □

Our second lemma is essentially due to Hauseux and Le Roux.

Lemma 5.5. *Let f be an automorphism of a complex surface with a wandering saddle configuration. Then $\mathrm{h_{pol}}(f) \geq 2$, and $\mathrm{h_{pol}}(f) = 2$ if f is an element of $\mathsf{Aut}(X)^0$.*

Sketch of proof. Take a point x_1 in $\mathsf{U}_1 \cap W^s(x)$ and a point x_2 in $W^u(x)$. Fix $\varepsilon > 0$ such that, for $i \in \{1,2\}$, the ball of radius ε centered at x_i is contained in U_i and is at distance $> \varepsilon$ from the complement of U_i. If ε is small enough, these balls are wandering. If ℓ is large enough, one can find a point z_ℓ in $B(x_1,\varepsilon)$ whose orbit $f^n(z_\ell)$ stays in the complement of the two balls except for $f^0(z_\ell) \in B(x_1,\varepsilon)$ and $f^\ell(z_\ell) \in B(x_2,\varepsilon)$. Then, the points $f^{-j}(z_\ell)$ for $j \leq n$ and $\ell \leq n/2$ are $(\varepsilon/2,n)$-separated (see [18], Example 2); the size of this set grows quadratically with n, hence $\mathrm{h_{pol}}(f) \geq 2$. The equality follows from $\mathrm{h_{pol}}(f) \leq 2$ when $f \in \mathsf{Aut}(X)^0$ (see § 5.1.1). □

5.2. **Irrational, ruled surfaces.** In this section, X is a ruled surface, but X is not rational. This means that there is a fibration $\pi\colon X \to B$ onto a base B of genus ≥ 1 with generic fiber $\mathbb{P}^1$. This fibration is equivariant with respect to $f\colon X \to X$ and an automorphism $f_B\colon B \to B$.

5.2.1. Assume, first, that f_B is periodic. Since the polynomial entropy does not change if one replace f by some positive iterate, we may as well suppose that $f_B = \mathsf{Id}_B$. The following lemma does not require that the genus of B be positive.

Lemma 5.6. *Let $\pi\colon X \to B$ be a ruled surface, and let f be an automorphism of X preserving each fiber. Then $\mathrm{h_{pol}}(f) \in \{0,1,2\}$, $\mathrm{h_{pol}}(f) = 0$ if and only if f preserves a kähler form, and $\mathrm{h_{pol}}(f) = 2$ if and only if f has a wandering saddle configuration.*

Sketch of Proof. First, assume that every fiber of π is a smooth rational curve. Then, the fibration is locally trivial: locally (X,π) is biholomorphically equivalent to $(\mathbb{D} \times \mathbb{P}^1(\mathbf{C}), \pi_1)$. Using coordinates $(x,[y_0:y_1])$ on $\mathbb{D} \times \mathbb{P}^1(\mathbf{C})$, the automorphism f can be written $f(x,[y_0:y_1]) = (x, A_x[y_0:y_1])$ for some holomorphic map $x \in \mathbb{D} \mapsto A_x \in \mathsf{Aut}(\mathbb{P}^1(\mathbf{C}))$. Lemma 5.4 shows that $\mathrm{h_{pol}}(f) \in \{0,1\}$, with $\mathrm{h_{pol}}(f) = 0$ if and only if A_x does not depend on x and is an isometry. Gluing the local charts, one sees that the same result holds globally for f on X.

Now, if X has singular fibers, X is not relatively minimal: it comes from a minimal model X_0 by blowing up (a finite sequence of) points, and f comes from an automorphism f_0 of X_0 acting trivially on B. Two cases may appear. One of the blow-ups creates a wandering saddle configuration: this occurs if one blows up a fixed point q at which Df_q has an eigenvalue α of modulus $\neq 1$. In that case $\mathrm{h_{pol}}(f) = 2$. Otherwise, $\mathrm{h_{pol}}(f) \in \{0,1\}$, and $\mathrm{h_{pol}}(f) = 0$ if and only if its iterates are contained in a compact group (a property that does not change from f_0 to f). □

5.2.2. Now assume that f_B has infinite order. Then, since $g(B) \geq 1$, we know that B is an elliptic curve. In that case, f is the flow, at time $t = 1$, of some holomorphic vector field on X that is transverse to the fibration π. All orbits of f are infinite, and in particular all fibers of π are smooth. The fibration falls in one of the following four types.

Example 5.7. Up to a finite base change, X is just the product $B \times \mathbb{P}^1(\mathbf{C})$ and $f(x,y) = (x+\tau, A(y))$ for some translation τ and some homography A.

Example 5.8. There are two sections σ_0 and σ_∞ of the fibration. If one removes them, the complement is isomorphic to the quotient of $\mathbf{C}^* \times \mathbf{C}^*$ by the action of a cyclic subgroup, acting by $(x,y) \mapsto (\lambda x, \mu y)$, with $|\lambda| < 1$ and $B = \mathbf{C}^*/\langle\lambda\rangle$. The action of f on X lifts to a diagonal transformation $F(x,y) = (\alpha x, \beta y)$ on $\mathbf{C}^* \times \mathbf{C}^*$.

Example 5.9. There is a unique section, when one removes it, the complement is isomorphic to the quotient of $\mathbf{C}^* \times \mathbf{C}$ by $(x,y) \mapsto (\lambda x, y+1)$ and f lifts to $F(x,y) = (\alpha x, y+\beta)$.

Example 5.10. There is no section, but there is one double section. This case reduces to the previous ones by a finite base change.

One easily gets the following result.

Lemma 5.11. *Either f is contained in a compact subgroup K of $\mathsf{Aut}(X)^0$, or the α and ω-limit set of every orbit is contained in a section of π; in that case,* $\mathrm{h_{pol}}(f) = 1$.

5.3. **Linear case.** Before studying rational surfaces in full generality, we focus on the linear projective case, i.e. automorphisms of the plane $\mathbb{P}^2(\mathbf{C})$.

Proposition 5.12. *Let g be an element of $\mathsf{PGL}_3(\mathbf{C}) = \mathsf{Aut}(\mathbb{P}^2(\mathbf{C}))$. Then* $\mathrm{h_{pol}}(g) \in \{0,1,2\}$. *More precisely, the following classification holds:*

(1) $\mathrm{h_{pol}}(g) = 0$ *if and only if g is an isometry;*
(2) $\mathrm{h_{pol}}(g) = 2$ *if and only if g has a wandering saddle configuration, if and only if g is conjugated to*

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{pmatrix} \tag{5.2}$$

with $|\lambda| > 1 > |\mu|$ *and* $|\nu| \neq 1$.

(3) $\mathrm{h_{pol}}(g) = 1$ *and* g *is conjugated*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \nu \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} \beta & 1 & 0 \\ 0 & \beta & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

with $|\nu| \neq 1$ *and* $|\alpha| = |\beta| = 1$.

Proof. Our goal is to prove the proposition, but also to introduce two technics: the blow up of fixed points, and the symbolic coding of Hauseux and Le Roux. Denote by $[x : y : z]$ the homogeneous coordinates of the plane.

The first step is to show that every linear projective transformation $g \in \mathsf{Aut}(\mathbb{P}^2(\mathbf{C}))$ is contained in one of the mentionned conjugacy classes. This is classical. Then, when g falls in case (1) it preserves a kähler metric, and its polynomial entropy vanishes. When g falls in case (2), one verifies that g has a wandering saddle configuration: in the diagonal case, on can take the fixed point $q = [0 : 0 : 1]$ and the stable and unstable varieties $\{y = 0\}$ and $\{x = 0\}$; in the second case one can take $q = [0 : 1 : 0]$ and the stable and unstable variety $\{z = 0\}$ and $\{x = 0\}$ (assuming $|\nu| < 1$).

Let us now look at case (3). First, assume that g is (conjugate to) a diagonal transformations with eigenvalues 1, α and ν with $|\alpha| = 1 < \nu$, i.e. $g(x, y) = (\frac{1}{\nu}x, \frac{\alpha}{\nu}y)$ in affine coordinates (x, y). Blow up the fixed point $[0 : 0 : 1]$ to get a new surface X on which g lifts to an automorphism g_X: the surface X fibers on $\mathbb{P}^1(\mathbf{C})$ (each fiber is the strict transform of a line through $[0 : 0 : 1]$, the action of g_X preserves this ruling, it acts by a rotation $w \mapsto \alpha w$ on the base, and as a loxodromic isometry in the fibers. One easily verifies (as for a product) that $\mathrm{h_{pol}}(g_X) = 1$. Since g has a wandering point we obtain $1 \leq \mathrm{h_{pol}}(g) \leq \mathrm{h_{pol}}(g_X) = 1$, hence $\mathrm{h_{pol}}(g) = 1$.

Assume now that g is (conjugate to) the Jordan block in case (3). Then g has a unique fixed point $q = [1 : 0 : 0]$ and all other points are wandering (their α and ω limit sets coincide with $\{q\}$). This setting has been studied by Hauseux and Le Roux in [18] and we can directly apply their result. Let X_0 be the complement of the fixed point q.

Let $\mathcal{F} = \{F_1, \dots F_k\}$ be a finite family of non-empty subsets of X_0. Let F_∞ be the complement of $\cup_{F_i \in \mathcal{F}} F_i$. To each orbit $(g^n(x))$, one associates its possible coding, i.e. the sequences of indices $i(n) \in \{1, \dots, k, \infty\}$ such that $g^n(x) \in F_{i(n)}$ for all $n \in \mathbf{Z}$ (the coding is not unique since the F_i may overlap). Let $Cod(N)$ be

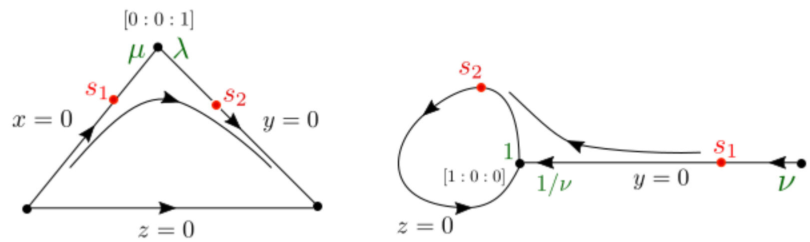

FIGURE 2. The dynamics of the maps described by the matrices in the point (2) of Proposition 5.12 is represented schematically. In both cases the singular sets giving the maximal local entropy contain two points on the separatrices of fixed points, $S = \{s_1, s_2\}$, and the times of passage from the neighbourhood of s_1 to the neighbourhood of s_2 can be arbitrary big.

the number of codes of length N which are realized by orbits of g; the polynomial degree growth of $Cod(N)$ is denoted $\mathrm{h_{pol}}(f;\mathcal{F})$. Then, one can define the **local polynomial entropy of g at a finite subset $\mathcal{S} \subset X_0$** as the limit of a (decreasing) sequence $\mathrm{h_{pol}}(f;U(\mathcal{S}))$ where $U(\mathcal{S})$ is the collection of open neighborhoods of points in $\mathcal{S}$ of decreasing size (see [18]). This number is denoted by $\mathrm{h_{pol}}^{\mathrm{loc}}(f;\mathcal{S})$. One says that subsets $U_1,\dots,U_L \subset X_0$ are **mutually singular** if for every M there exists a point $x \in X$ and times $n_1,\dots,n_L$ such that $g^{n_i}(x) \in U_i$ for any i and $|n_i - n_j| > M$ for every $i \neq j$. A finite set $\{s_1,\dots,s_n\} \in X_0$ is **singular** if all small enough neighborhoods $U_1,\dots,U_L$ of $s_1,\dots,s_L$ respectively are mutually singular. Any singleton is a singular set. Hauseux and Le Roux prove that $\mathrm{h_{pol}}(f)$ can be calculated as a following supremum:

$$\mathrm{h_{pol}}(g) = \sup\left\{\mathrm{h_{pol}}^{\mathrm{loc}}(f;\mathcal{S}) \mid \mathcal{S} \text{ is a singular set}\right\}. \tag{5.3}$$

Now, in the example of the Jordan block, one sees that every singular set is reduced to a singleton (one can, for instance, take a point that is not fixed on the unique g-invariant line). This shows that $\mathrm{h_{pol}}(g) = 1$. □

Remark 5.13. Note that the Proposition 5.12, and its proof can be repeated word by word for $\mathsf{PGL}_3(\mathbf{R})$.

5.4. **Rational surfaces.** To prove Theorem 5.1 it remains to study rational surfaces which are not isomorphic to the projective plane. First, we study minimal, rational surfaces and then we look at how polynomial entropy changes under blow-ups and use the following

Lemma 5.14. *Let* X, Y *be complex Kähler manifolds,* $f \in \mathsf{Aut}(Y), g \in \mathsf{Aut}(X)$ *and the projection* $\pi : Y \to X$ *is such that* π *is* 1 *to* 1 *everywhere except for* q *and that* π^{-1} *blows up* q*, a fixed point of* g *and* $\pi^{-1}(q) = K$ *is a compact in* Y*. Suppose that* $\pi \circ f = g \circ \pi$ *and that* $f|_K$ *is an isometry on* K *and that* $\mathrm{h_{top}}(f) = 0$*. Then* $\mathrm{h_{pol}}(f) = \mathrm{h_{pol}}(g)$*.*

5.4.1. *Minimal rational surfaces.* Let X be a minimal rational surface, and assume that X is not the projective plane. When X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, we know from Example 5.2 that the conclusion of Theorem 5.1 is satisfied. Thus, we assume that X is not isomorphic to $\mathbb{P}^1(\mathbf{C}) \times \mathbb{P}^1(\mathbf{C})$ or $\mathbb{P}^2(\mathbf{C})$. Then, there is a unique ruling $\pi \colon X \to \mathbb{P}^1(\mathbf{C})$ invariant under the action of $\mathsf{Aut}(X)$, with a unique section $C \subset X$, of negative self intersection.

Consider the automorphism f_B of the base $B = \mathbb{P}^1(\mathbf{C})$ of the ruling that is induced by f. By Lemma 5.6 we may assume that f_B is not the identity, hence it is either elliptic, parabolic, loxodromic. Take homogeneous coordinates on B such that the fixed point are $\{0, \infty\}$ or $\{0\}$ if f_B is parabolic. One can trivialize the fibration above a neighborhood U of 0 in such a way that C corresponds to the points at infinity in the fiber and $f(x,y) = (f_B(x), y + q(x))$ for some rational function $q(x)$. If there is a wandering saddle configuration, $\mathrm{h_{pol}}(f) = 2$. If not, we see that f_B is elliptic or $q(0) = 0$ (and similarly $q(\infty) = 0$). In the first case, it is easy to see that $\mathrm{h_{pol}}(f) = 1$ except when q vanishes identically in which case f is contained in a compact group and $\mathrm{h_{pol}}(f) = 0$. In the second case, one also get $\mathrm{h_{pol}}(f) = 1$ (every orbit converges towards a fixed point above 0 or ∞ along an f-invariant algebraic curve).

5.4.2. *Non minimal rational surfaces.* There is a minimal rational surface X_0, an automorphism f_0 of X_0 and a birational morphism $\tau \colon X \to X_0$ such that $\tau \circ f = f_0 \circ \tau$. The inverse mapping τ^{-1} blows up fixed points of f_0. If one blows up a point q such that Df_q has two eigenvalues of distinct moduli, one finds a wandering saddle configuration and $\mathrm{h_{pol}}(f) = 2$ (and hence, the entropy can possibly grow by blow-up). If not, $\mathrm{h_{pol}}(f) = \mathrm{h_{pol}}(f_0)$.

This concludes the proof of Theorem 5.1.

5.5. **Polynomial entropy for surfaces.**

5.5.1. *Hodge decomposition.* Fix a Kähler form κ on X, and denote by $[\kappa] \in H^2(X; \mathbf{R})$ its cohomology class. First, recall the Hodge decomposition

$$H^n(X, \mathbf{C}) \cong \bigoplus_{p+q=n} H^{p,q}(X, \mathbf{C}) \tag{5.4}$$

where $H^{p,q}(X,\mathbf{C})$ is the subspace of cohomology classes of type (p,q). This decomposition is invariant under the action of $\mathrm{Aut}(X)$. Moreover, κ determines an intersection form Q_κ on $H^2(X,\mathbb{R})$:

$$Q_\kappa([\alpha],[\beta]) = [\alpha]\cup[\beta]\cup[\kappa]^{k-2} = \int_X \alpha\wedge\beta\wedge\kappa^{k-2}. \tag{5.5}$$

Theorem 5.15 (Hodge index theorem). *The restriction to* $H^{1,1}(X,\mathbf{C})$ *of the Hermitian product*

$$Q_\kappa : H^2(X,\mathbf{C})\times H^2(X,\mathbf{C}) \to \mathbf{C}$$
$$([\alpha],[\beta]) \mapsto \int_X \alpha\wedge\bar{\beta}\wedge\kappa^{k-2}$$

associated to the intersection form Q_κ *has signature* $(1,h^{1,1}(X)-1)$ *where* $h^{1,1}(X) = \dim H^{1,1}(X,\mathbf{C})$. *The restriction of this Hermitian product* Q_κ *to* $H^{2,0}(X,\mathbf{C})\oplus H^{0,2}(X,\mathbf{C})$ *is positive definite.*

5.5.2. *Surfaces.* Assume that X is a surface: $k=2$, and Q_κ is denoted Q because it does not depend on κ; the signature of Q on $H^{1,1}(X;\mathbf{R})$ being equal to $(1,h^{1,1}(X)-1)$, it determines a structure of Minkowski space on $H^{1,1}(X;\mathbf{R})$. The classification of isometries of Minkowski spaces and the geometry of surfaces lead to the following three cases (see [7]):

(1) f^* is an elliptic isometry of $H^{1,1}(X;\mathbf{R})$, and then there exists a positive iterate f^m of f such that $f^m\in\mathsf{Aut}(X)^0$. In particular, there is a holomorphic vector field θ on X such that f^m is the flow, at time 1, obtained by integrating θ: $f^m(x) = \Phi_\theta(1,x)$.

(2) f^* is a parabolic isometry of $H^{1,1}(X;\mathbf{R})$. In that case, there exists a fibration $\pi\colon X\to B$ onto a Riemann surface B whose generic fibers are connected and of genus 1, which is invariant under the action of f: there is an automorphism f_B of B such that $\pi\circ f = f_B\circ\pi$. Moreover, the growth of $(f^n)^*$ on $H^{1,1}(X;\mathbf{R})$ is quadratic: $\parallel (f^n)^* \parallel\approx Cn^2$ for some positive constant C.

(3) f^* is a loxodromic isometry of $H^{1,1}(X;\mathbf{R})$, f^* has a unique eigenvalue of modulus >1 on $H^*(X;\mathbf{C})$, this eigenvalue coincides with $\lambda(f)$, and it is realized on $H^{1,1}(X;\mathbf{R})$. The topological entropy of f is positive, and equal to $\log(\lambda(f))$.

Moreover, in the loxodromic case, $f\colon X\to X$ has infinitely many saddle periodic points, and these periodic points equidistribute towards the unique f-invariant probability measure of maximal entropy (see [5, 7, 14]).

Theorem 5.16. *Let f be an automorphism of a compact Kähler surface. If f^* is elliptic,* $\mathrm{h}_{\mathrm{pol}}(f) \in \{0,1,2\}$*. If f^* is parabolic, then* $\mathrm{h}_{\mathrm{pol}}(f) \in [2,4]$ *and is at most* 4*. If f^* is loxodromic,* $\mathrm{h}_{\mathrm{pol}}(f) = \infty$ *since* $\mathrm{h}_{\mathrm{top}}(f) = \log(\lambda(f))$ *is positive.*

Proof. The first statement is a consequence of Theorem 5.1.

We know from Gromov's upper bound that $\mathrm{h}_{\mathrm{pol}}(f) \leq 4$ when f is parabolic. (see Section 4.1) Also, in that case, f preserves a fibration $\pi \colon X \to B$ by curves of genus 1 (except finitely many singular fibers). If the action on the base $f_B \in \mathsf{Aut}(B)$ has finite order, we can assume $f_B = \mathsf{Id}$. Then, if $b \in B$ is a regular value of π, there is a local $\mathcal{C}^\infty$ change of coordinates near $\pi^{-1}(b)$ that conjugates f to a twist map $(x_1,x_2,y_1,y_2) \mapsto (x_1,x_2,y_1+t_1(x_1,x_2),y_2+t_2(x_1,x_2))$ with (y_1,y_2) in the torus $\mathbf{R}^2/\mathbf{Z}^2$; moreover, $(x_1,x_2) \mapsto (t_1,t_2)$ is locally dominant (see [6]). This implies that the polynomial entropy of f in this invariant neighborhood of $\pi^{-1}(b)$ is 2. If the action on the base has infinite order then f is an affine transformation of a complex torus whose linear part is parabolic and $\mathrm{h}_{\mathrm{pol}}(f) = 2$ (see [7]). □

Part III.– Minimal actions and Zariski dense orbits, the surface case.

6. AUTOMORPHISMS WITH NO FINITE ORBITS

In this section, we classify automorphisms of surfaces satisfying one of the following properties

- all orbits of f are infinite, i.e. $f\colon X \to X$ has no finite orbit;
- all orbits of f are Zariski dense;
- all orbits of f are dense for the euclidean topology.

Those properties are listed from the weakest to the strongest: euclidean density implies Zariski density, which in turn excludes the existence of periodic orbit. The last property is exactly the notion of minimality (with respect to the euclidean topology).

6.1. **Automorphisms with no finite orbits.**

6.1.1. *The Lefschetz formula and the Albanese morphism.* Let f be an automorphism of a compact Kähler surface without periodic orbits.

Lemma 6.1. *The action of f on the cohomology of X is virtually unipotent.*

Proof. If there is an eigenvalue $\lambda \in \mathbf{C}$ of f^* with $|\lambda| > 1$, we know from [7] that this eigenvalue is unique and has multiplicity 1. Thus, the alternating sum of the traces of $(f^n)*$ on the cohomology groups $H^k(X;\mathbf{R})$ grows like λ^n, and the Lefschetz formula shows that f has at least one finite orbit (in fact it has infinitely many saddle periodic points, see [7]). We deduce that all eigenvalues of f^* have modulus ≤ 1. Since f^* preserves the integral cohomology $H^*(X;\mathbf{Z})$, its eigenvalues are algebraic integers, and Kronecker lemma shows that they are roots of unity. We deduce that some positive iterate $(f^n)^*$ is unipotent. □

Now, choose $n > 0$ such that $(f^n)^*$ is unipotent; since f^n has no fixed point, the holomorphic Lefschetz formula gives $h^{2,0}(X) - h^{1,0} + 1 = 0$ (see [7]):

Lemma 6.2. *If there is an automorphism of X without periodic orbit, then X satisfies*

$$h^{1,0}(X) = h^{2,0}(X) + 1.$$

In particular, there are non-trivial holomorphic 1-forms on the surface X, and X is not a rational surface.

Since $h^{1,0}(X)$ is positive, the Albanese map determines a non-trivial morphism $\alpha_X: X \to A_X$, where $A_X = H^0(X,\Omega^1_X)/H_1(X;\mathbf{Z})$ is the Albanese torus. This map is equivariant with respect to the action of f on X, and its induced action $f_{alb}: A_X \to A_X$; this means that $f_{alb} \circ \alpha_X = \alpha_X \circ f$.

6.1.2. *Invariant genus* 1 *pencil.* Suppose that f^* is unipotent and not equal to the identity. Then f is parabolic and preserves a unique genus 1 fibration $\pi: X \to B$ onto some Riemann surface B; this means that there is an automorphism f_B of B such that $f_B \circ \pi = \pi \circ f$ (see Section 5.5.2). Moreover, either f_B is periodic, or the surface X is a torus (see [7]). In particular, if all orbits of f are Zariski dense, then X must be a torus.

Assume that f_B is periodic, and replace f by f^n where n is the order of f on the base. Given $b \in B$, f preserves the fiber $X_b = \pi^{-1}(b)$. If X_b is not a smooth curve of genus 1, then there is a periodic orbit of f in X_b. Thus, if all orbits of f are infinite, then all fibers of $\pi: X \to B$ are smooth curves of genus 1 (some of them may a priori be multiple fibers).

6.1.3. *Tori and bi-elliptic surfaces.* We consider two special cases, namely

(1) the minimal model of X is a torus;
(2) the minimal model of X is a bi-elliptic surface.

Let $\pi: X \to X_0$ be the birational morphism onto the minimal model X_0 of X. The exceptional divisor of π coincides with the vanishing locus of all holomorphic 2-forms of X. This implies that f preserves this divisor, and some iterate f^n fixes each of its irreducible components. Those components being rational curves, f^n has a fixed point on each of them, contradicting the absence of periodic orbits. Thus, X coincides with X_0.

If X is a bi-elliptic surface, it is the quotient of an abelian surface $A = B \times C$, where B and C are two elliptic curves, by a finite group G acting diagonally on A: the action on B is by translation $x \mapsto x + \varepsilon$, and the action on C is of the form

$$y \mapsto \omega y + \eta\,, \tag{6.1}$$

where ω is a root of 1 of order 2, 4, 3 or 6. The automorphism f of X lifts to an automorphism $\tilde{f}$ of the universal cover $\mathbf{C}^2$; here $\mathbf{C}^2 = \mathbf{C} \times \mathbf{C}$, with coordinates (x,y), and the elliptic curves B and C are the quotients of the x-axis and the y-axis by lattices Λ_B and Λ_C. Write $\tilde{f}(x,y) = L(x,y) + (a,b)$ for some linear transformation $L \in GL_2(\mathbf{C})$. Since $\tilde{f}$ covers f, the linear map normalizes the linear part $(x,y) \mapsto (x,\omega y)$ of G. Thus, L is a diagonal matrix. But from Lemma 6.1, it is also virtually unipotent. We deduce that $L^n = \mathsf{Id}$ for some

$n > 0$. Changing f in f^{kn} for some $k > 0$, we may assume that f is covered by a translation that commutes to the linear part of G. Thus, *some positive iterate f^m of f is covered by a translation of type* $(x,y) \mapsto (x+a,y)$. This proves the following lemma.

Lemma 6.3. *Let f be an automorphism of a complex projective surface X with no finite orbit. If the minimal model of X is bi-elliptic, then X coincides with its minimal model, and a positive iterate of f is covered by a translation $(x,y) \mapsto (x+a,y)$ of a product $B \times C$ of two elliptic curves. In particular, the Zariski closure of each orbit is a curve of genus* 1.

Let us now study the case of tori.

Example 6.4. Let f be a translation on a 2-dimensional compact torus $X = \mathbf{C}^2/\Lambda$, and let M be the closure of the orbit of the neutral element $(0,0) \in X$. Then M is a real Lie subgroup of X; its connected component of the identity is a real torus $M^0 \subset X$. The orbit $\{f^n(x); n \in \mathbf{Z}\}$ of any point $x \in X$ is dense in $x+M$.

Thus, on every compact torus, there are examples of translations whose orbits are Zariski dense but not dense for the euclidean topology.

Example 6.5 (Furstenberg [15])**.** Consider a 2-dimensional torus X which is the product of two copies of the same elliptic curve E; write $E = \mathbf{C}/\Lambda$ and $X = \mathbf{C}^2/(\Lambda \times \Lambda)$ for some lattice $\Lambda \subset \mathbf{C}$. Then, consider the automorphism

$$f(x,y) = (x+a, y+x+b) \tag{6.2}$$

for some pair of elements $(a,b) \in E \times E$. Assume that a is totally irrational with respect to $\mathbf{C}$, i.e. $x \mapsto x+a$ has dense orbits in E for the euclidean topology. Then, all orbits of f are dense for the euclidean topology.

Lemma 6.6. *There are examples of automorphisms f on abelian surfaces such that*

- *f is elliptic, all orbits of f are Zariski dense, but no orbit is dense for the euclidean topology;*
- *f is elliptic and all orbits of f are dense for the euclidean topology;*
- *f is parabolic, and all orbits of f are dense for the euclidean topology.*

If f is an automorphism of an abelian surface with no finite orbit, then the following are equivalent

- *one orbit of f is dense for the Zariski topology (resp. for the euclidean topology);*

- *every orbit of f is dense for the Zariski topology (resp. for the euclidean topology);*

Proof. We only have to prove the second assertion. So, assume that there is an orbit of f that is dense for the euclidean topology; we want to prove that all orbits are dense. If some positive iterate of f is a translation, this is easy. If not, f is parabolic, and we may assume that $x = E \times E$, with

$$f(x,y) = (x+a, y+kx) \tag{6.3}$$

for some $k \geq 1$. Then, $x \mapsto x+a$ has dense orbits. If there is one orbit which is not dense, then there is a non-trivial minimal invariant subset M in $E \times E$. But this is impossible by Furstenberg's results.

Now, assume that there is an orbit of f that is dense for the Zariski topology, and that f is parabolic. Again, $X = E \times E$, and we can assume that f is as in Equation (6.3), with $x \mapsto x+a$ a translation of infinite order. If the orbit of (x,y) is not Zariski dense, then its Zariski closure is an f-invariant curve $C \subset E \times E$, on which f induces an automorphism of infinite order. Thus, C is a curve of genus 1, embedded in an f-invariant way into $E \times E$. But then, the translates of C form an f-invariant pencil, and since f is parabolic, this pencil must coincide with the unique f-invariant fibration $(x,y) \mapsto x$. We get a contradiction because a is not a torsion point of E. □

6.1.4. *Ruled surfaces (first step).* Let us now assume that the Kodaira dimension of X is $-\infty$. The Albanese map provides a fibration $\alpha_X \colon X \to A_X$, where $A_X = \mathbf{C}/\Lambda$ is a curve of genus 1. The automorphism f induces an automorphism f_{alb} of A_X. If $f_{alb}^n(x) = x$ for some $m > 0$, the fiber $\alpha_X^{-1}(x)$ is a curve of genus 0 (it may be singular), and f^m must fix a point in this fiber. Thus, the absence of finite orbit for f implies that all orbits of f_{alb} are infinite, and f_{alb} is a translation of A_X with Zariski dense orbits. As a consequence, α_X is a submersion and X is a fiber bundle over A_X with rational fibers. The action of f^* on $H^{1,1}(X;\mathbf{R})$ can not be parabolic, because in that case f preserves a unique fibration and this fibration is by curves of genus 1. Thus, some positive iterate of f is an element of $\mathsf{Aut}(X)^0$: there is a holomorphic vector field θ on X and f is the flow of θ at time $t = 1$. This flow must permute the fibers of α_X and it is transverse to the fibration.

Lemma 6.7. *Let f be an automorphism of a complex projective surface without finite orbits. If $\mathsf{kod}(X)$ is negative the albanese map $\alpha_X \colon X \to A_X$ is a* *submersion onto an elliptic curve whose fibers are rational curves. Some positive iterate f^m of f is the flow, at time 1, of a vector field which is everywhere transverse to the fibration.*

Consider the monodromy of the foliation induced by this vector field: it gives a representation of $\pi_1(A_X;x)$ into $\mathsf{Aut}(\alpha_X^{-1}(x))$, i.e. into the group $PGL_2(\mathbf{C})$ of automorphisms of $\mathbb{P}^1$. Since $\pi(A_X;x) \simeq \mathbf{Z}^2$, the monodromy group has a fixed point p. The orbit of p under the flow of θ is a section of the fibration and is invariant under the action of f^m. Thus *at least one orbit of f^m is contained, and Zariski dense, in an elliptic curve.*

6.1.5. *Zariski dense orbits.*

Theorem 6.8. *Let f be an automorphism of a compact Kähler surface. If all orbits of f are Zariski dense, then X is a torus, and on every torus there are translations whose orbits are Zariski dense (resp. dense for the euclidean topology).*

Proof. If the Kodaira dimension of X is non-negative, some positive multiple mK_X of the canonical bundle has non-trivial sections. Fix such a multiple, and consider the action of f on the space of sections $H^0(X;mK_X)$. The existence of an eigenvector provides a section ω of mK_X such that $f^*\omega = \xi\omega$ for some $\xi \in \mathbf{C}^*$. In particular, the vanishing locus of ω is either empty, or an f-invariant curve. Since all orbits of f are Zariski dense, f does not preserve any curve and ω does not vanish: this proves that mK_X is the trivial bundle and that $\mathsf{kod}(X) = 0$.

If the Kodaira dimension of X is negative, Lemma 6.7 shows that f has an orbit which is contained in a finite union of curves of genus 1, contradicting our hypothesis. Thus $\mathsf{kod}(X) = 0$.

Now, since $\mathsf{kod}(X) = 0$ and $h^{1,0}(X) > 0$ (see Lemma 6.2), the minimal model of X is a torus or a bi-elliptic surface, and we conclude with Lemma 6.3. □

6.1.6. *Ruled surfaces (second step).* Let us come back to the study of ruled surfaces $\alpha_X : X \to A_X$ with an automorphism f whose orbits are all infinite. According to Section 6.1.4, we can assume that $f \in \mathsf{Aut}(X)^0$ and f is the flow of a vector field θ that is transverse to the fibration α_X.

To simplify the notation, denote by G the group $\mathsf{Aut}(X)^0$. By the universal property of the Albanese morphism, there is an α_X-equivariant action of G on A_X; and this action factors through the Abanese torus, via a homomorphism $A_G \to A_X$. Since the flow Φ_θ^t provides a non-trivial flow on A_X, we deduce that

G acts transitively on A_X. If $\dim(G) \geq 2$ the dimension of the kernel H of the homomorphism $G \to A_X$ is positive, H has positive dimensional orbits in the fibers of α_X, and X is almost homogeneous: the group G has an open orbit (for the Zariski topology). We shall study this case below.

If $\dim(G) = 1$ then G is isogeneous to A_X: G is an elliptic curve, and there is a homomorphism $G \to A_X$ whose kernel $F \subset G$ is finite; moreover, the general orbit of G in X is isomorphic to G, and the action of F on the base A_X of α_X is trivial. Moreover, a theorem of Nishi and Matsumura asserts that X is isomorphic to a suspension $G \times^F \mathbb{P}^1_{\mathbf{C}}$, where F acts on $\mathbb{P}^1_{\mathbf{C}}$ via the homomorphism $\rho \colon F \to PGL_2(\mathbf{C})$ giving the action of F on the fibers of α_X. Since F is a finite abelian group, $\rho(F)$ is a finite, cyclic, diagonalizable subgroup of $PGL_2(\mathbf{C})$; we can write $\rho(a)[x:y] = [\xi(a)x : y]$ where $\xi(a)$ is a root of unity that depends on $a \in F$. In these coordinates X is the quotient of $G \times \mathbb{P}^1_{\mathbf{C}}$ by the action

$$(z,[x:y]) \in G \times \mathbb{P}^1_{\mathbf{C}} \mapsto (z+a, [\xi(a):y]) \tag{6.4}$$

of the finite group $F \subset G$. But then, the transformations $(z,[x:y]) \mapsto (z+s,[\mu x : y])$ commute to the action of F for every pair $(s,\mu) \in G \times \mathbf{C}^*$, and they induce a group of automorphisms of dimension ≥ 2 on X, contradicting $\dim(G) = 1$. Thus, $\dim(G) \geq 2$ and X is almost homogeneous.

To complete our study, we now rely on Section 3 of [35] (namely the constructions on pages 251–253). Since X is a ruled, almost homogeneous surface, X is a topologically trivial $\mathbb{P}^1_{\mathbf{C}}$-bundle over the elliptic curve A_X, and there are two types of such bundles:

(a) X is the quotient of $\mathbf{C}^* \times \mathbb{P}^1_{\mathbf{C}}$ by the automorphism $(z,[x:y]) \mapsto (\lambda z, [\mu x : y])$ for some pair (λ,μ) of complex numbers with $\lambda\mu \neq 0$ and $|\lambda| < 1$.
(b) X is the quotient of $\mathbf{C}^* \times \mathbb{P}^1_{\mathbf{C}}$ by the automorphism $(z,[x:y]) \mapsto (\lambda z, [x + y : y])$ for some complex number λ with $|\lambda| < 1$.

Case (a) correspond in fact to two subcases. If μ is equal to 1, then X is just the product $A_X \times \mathbb{P}^1_{\mathbf{C}}$, and then every automoorphism is of the form $f \colon (z,[x:y]) \mapsto (u(z), v[x:y])$ where u is an automorphism of A_X and v is an element of $PGL_2(\mathbf{C})$. *No orbit of f is dense for the euclidean topology; the general orbit of f is Zariski dense if and only if u and v are two automorphisms of infinite order.* If μ is a root of unity, the same result holds. Then, assume that μ is not a root of unity. Using affine coordinates $x = [x:1]$ for $\mathbb{P}^1_{\mathbf{C}}$, one sees that every automorphism of X comes from an automorphism of $\mathbf{C}^* \times \mathbf{C}^*$ of type $(z,x) \mapsto (\alpha z, \beta x)$ or $(\alpha z^{-1}, \beta x)$ for some pair of complex numbers (α,β) with $\alpha\beta \neq 0$. Changing f in f^2 we assume that $f(z,x) = (\alpha z, \beta x)$. Then, *the general* *orbit of f is Zariski dense*; indeed, the action of f on A_X has infinite order because otherwise f has a periodic orbit, if the Zariski closure of a general orbit is not X, then it is a multi-section of the fibration α_X, but there are only two such multi-sections. A point (z_0, x_0) has a dense orbit for the euclidean topology if and only if for every point (z_1, x_1) and every $\varepsilon > 0$, there are integers m and n such that $(\alpha^n z_0, \beta^n x_0)$ is ε-close to $(\lambda^m z_1, \mu^m x_1)$ in $\mathbf{C}^* \times \mathbf{C}^*$. Taking logarithms, this means that the vectors $(\ln(\alpha), \ln(\beta))$, $(\log(\lambda), \log(\mu))$, $(2i\pi, 0)$, and $(0, 2i\pi)$ generate a dense subgroup of $\mathbf{C} \times \mathbf{C}$, which of course is impossible since the rank of this group is at most 4. This argument shows that in case (a), *there is no automorphism with a dense orbit for the euclidean topology*.

In case (b), every automorphism of X can be written $(z, x) \mapsto (\alpha z, x + \beta)$ and, again, *the general orbit of f is Zariski dense, but no orbit is dense for the euclidean topology*.

6.1.7. *Conclusion.*

Theorem 6.9. *Let f be an automorphism of a compact Kähler surface X, all of whose orbits are infinite. Replacing f by some positive iterate, there are only three possibilities:*

(1) $\mathsf{kod}(X) = 1$*, X is a fibration over a curve B with smooth fibers of genus* 1 *(some of them can be multiple fibers), and every orbit of f is Zariski dense in such a fiber.*

(2) $\mathsf{kod}(X) = 0$*, and X is a torus or X is a bi-elliptic surface. If X is bi-elliptic, then the Zariski closure of every orbit is a curve of genus* 1*. If X is a torus, an orbit is dense for the Zariski (resp. for the euclidean) topology if and only if all orbits are dense for this topology.*

(3) $\mathsf{kod}(X) = -\infty$*, then X is a ruled surface over an elliptic curve, at least one orbit is contained, and dense, in an elliptic curve (a section of the ruling), but no orbit is dense for the euclidean topology. Either X is isomorphic to the quotient of $\mathbf{C}^* \times \mathbb{P}^1_{\mathbf{C}}$ by $(z, [x : y]) \mapsto (\lambda z, [\xi x : y])$ with $0 < |\lambda| < 1$ and ξ a root of unity, an iterate of f is of the form $(z, [x : y]) \mapsto (\alpha z, [\beta x : y])$ with β a root of unity; in this situation the general orbit is dense along a multi-section of the fibration α_X. Otherwise, the general orbits of f are dense in X.*

Proof. All we have to do, is put together the previous results of this section together with Enriques-Kodaira classification of surfaces. First, if the Kodaira dimension of a projective variety is maximal, its group of automorphisms is finite. Thus, we can assume $\mathsf{kod}(X) \leq 1$.

Assume that $\mathsf{kod}(X) = 1$. The action of f on the base of the Kodaira-Iitaka fibration is periodic (see [37]). Since every orbit of f is infinite, every fiber of this fibration is a smooth curve of genus 1, and every orbit is dense in such a fiber.

When $\mathsf{kod}(X) = 0$, we know that X must be a minimal surface, and that $h^{1,0}(X) \geq 1$. Thus, X is a torus or a bi-elliptic surface. Then, we refer to Lemma 6.3 and Lemma 6.6.

When $\mathsf{kod}(X) = -\infty$, we know from Section 6.1.4 that X is a ruled surface over an elliptic curve, and the conclusion follows from Section 6.1.6. □

6.2. **Open questions.** We wonder if the result of Theorem 6.8 can be generalized to higher dimensions.

Question 6.1. Let f be an automorphism of a complex projective manifold X of dimension 3 (or more) acting minimally on X. Is X automatically a torus ?

Consider the real manifold $M = \mathsf{SO}_3(\mathbf{R})$, of dimension 3, and pick a free group F_2 of rank 2 in $\mathsf{SO}_3(\mathbf{R})$; then, F_2 acts freely on M (by translations).

Question 6.2. Does there exist a complex projective manifold X of dimension 3 and a free subgroup of rank 2 in $\mathsf{Aut}(X)$ acting freely on X ?

Part IV.– Small entropy and degree growth

7. SMALL ENTROPY: GENERAL FACTS

In this section we gather a few remarks and examples concerning homeomorphisms of compact spaces with small polynomial entropy.

7.1. **Recurrence properties.**

Lemma 7.1. *Let f be a homeomorphism of a compact metric space X.*

(1) *If $\mathrm{h}_{\mathrm{pol}}(f) < 1$, then for every $\varepsilon > 0$, there exists $k > 0$ such that for every $x \in X$ there is a time $j \leq k$ with $dist(x, f^j(x)) \leq \varepsilon$.*
(2) *If the polynomial entropy of f is < 1, all points of X are recurrent, none of them is wandering.*
(3) *If the polynomial entropy of f is $< 1/2$, then $\limsup dist(f^n(x), f^n(y)) \geq dist(x,y)/2$ for every pair of points (x,y) in $X \times X$.*

Proof. The first part, due to [1, Prop. 2.1], is obtained as follows. Suppose there is $\varepsilon > 0$, such that for all $k > 0$ one can find a point y with $dist(y, f^j(y)) > \varepsilon$ for all $1 \leq j \leq k$. Set $x_j = f^j(y)$ for $0 \leq j \leq k$. Then, for the dynamics of f^{-1}, the points x_j are (ε, k)-separated, because if $j < j'$ the distance between $f^{-j}(x_j) = y$ and $f^{-j}(x_{j'}) = x_{j'-j}$ is greater than ε. Thus, $h_{pol}(f) \geq 1$.

The second assertion is a consequence of the first. The third follows from the second one, applied to $f \times f$. □

Remark 7.2. A point $x \in X$ is *uniformly recurrent* for a homeomorphism $f : X \to X$ of a compact metric space if for any $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for any $n \in \mathbf{N}$ among any succesive iterates $f^{n+k}(x), k = 0, \ldots, N-1$, there exists at least one such that $d(x, f^{n+k}(x)) < \varepsilon$. Lemma 7.1 does not say that all points are uniformly recurrent.

7.2. **Growth of derivatives.**

Lemma 7.3. *Let f be a homeomorphism of a compact manifold X. Denote by $Lip_f(n)$ the maximum of the lipschitz constants of Id, f, …, f^{n-1} ($Lip_f(n)$ is infinite if f is not lipschitz). Then, X is covered by $O_\varepsilon(Lip_n(f)^{\dim(X)})$ balls of radius $\leq \varepsilon$ for the iterated metric $dist_n$. If $Lip(n) \leq n^\alpha$ for some $\alpha > 0$, then $h_{pol}(f) \leq \alpha \dim(X)$; if $Lip(n) = o(n^\alpha)$ for all $\alpha > 0$, then $h_{pol}(f) = 0$.*

Proof. If x and y satisfy $dist(x,y) \leq \frac{1}{2}\varepsilon Lip_f(n)^{-1}$, then the distance between $f^k(x)$ and $f^k(y)$ is less than ε for every natural integer $k \leq n-1$. And one can cover X by roughly $(2\varepsilon^{-1} Lip_f(n))^{\dim(X)}$ balls of radius $\frac{1}{2}\varepsilon Lip_f(n)^{-1}$. □

Theorem 7.4. *Let f be a diffeomorphism of class $\mathcal{C}^2$ of a closed manifold M. Assume that the growth of the derivative of f^n is exponential: there is $\eta > 0$ such that $\| Df^n \| \geq \exp(\eta n)$ as n goes to $+\infty$. Then, $h_{pol}(f) \geq 1/2$.*

A diffeomorphism of the sphere with a north-south dynamics has polynomial entropy equal to 1; it would be good to replace the inequality $h_{pol}(f) \geq 1/2$ by $h_{pol}(f) \geq 1$ in this theorem.

Sketch of the Proof. From [4], there exists an f-invariant ergodic probability measure μ on M with a positive Lyapunov exponent. Pesin's theory implies that a μ-generic point x has a non-trivial unstable manifold. Let x and y be points of such an unstable manifold. Then the distance between $f^n(x)$ and $f^n(y)$ goes to 0 as n goes to $-\infty$; by Lemma 7.1, this shows that $h_{pol}(f) \geq 1/2$. □

7.3. **Skew products.** The following theorem answers a question of Artigue, Carrasco-Olivera, and Monteverde (see Problem 1 in [1]).

Theorem 7.5. *There exists an analytic, area preserving diffeomorphism f of the torus $\mathbb{T}^2$ satisfying the following four properties*

1. *f is minimal;*
2. *its iterates f^n, $n \in \mathbf{Z}$, do not form an equicontinuous family;*
3. *for every $\varepsilon > 0$, the norm of the derivative of f^n satisfies $\| Df^n \|_{\mathbb{T}^2} = o(n^\varepsilon)$;*
4. *the polynomial entropy of f vanishes.*

Remark 7.6. One can construct such examples on all tori $\mathbb{T}^k$, $k \geq 2$ but, if f is a homeomorphism of the circle with polynomial entropy 0, then f is conjugate to a rotation (see [26]).

Remark 7.7. Let σ be the shift on $\Lambda^{\mathbf{Z}}$ for some finite alphabet Λ. One easily proves that the polynomial entropy of a subshift $\sigma_K \colon K \to K$ is ≥ 1 for every σ-invariant infinite compact subset $K \subset \Lambda^{\mathbf{Z}}$. More generally, every expansive homeomorphism of an infinite compact metric space has polynomial entropy ≥ 1 (see [1]).

Remark 7.8. Fix a function $\varphi \colon \mathbf{R}_+ \to \mathbf{R}_+$ such that φ is increasing, φ is unbounded, φ does not vanish, and $\varphi(x) = o(x)$ as x goes to $+\infty$. By a result of Borichev (see [3], and also [34]), there is an analytic diffeomorphism f of $\mathbf{R}^2/\mathbf{Z}^2$ that preserves the Lebesgue measure and satisfies

$$Lip_f(n) \leq \varphi(n) \quad \text{and} \quad \limsup_{n \to +\infty} \frac{Lip_f(n)}{\varphi(n)} > 0. \tag{7.1}$$

In fact, one can construct such an f as a skew product $f(x,y) = (x+\alpha, y+g(x))$ for some well chosen periodic function g and angle α. The proof of Theorem 7.5 follows a similar strategy (and is simpler).

7.3.1. *Skew product.* Let $\mathbb{T}$ denote the circle $\mathbf{R}/\mathbf{Z}$, so that $\mathbb{T}^d = \mathbf{R}^d/\mathbf{Z}^d$ is the torus of dimension d. Let $\alpha \in \mathbf{R}/\mathbf{Z}$ be an irrational number, and let $g\colon \mathbb{T} \to \mathbf{R}$ be a continuous function such that $\int_0^1 g(x)dx = 0$. Consider the homeomorphism $f\colon \mathbb{T}^2 \times \mathbb{T}^2$ defined by

$$f(x,y) = (x+\alpha, y+g(x)). \tag{7.2}$$

The n-th iterate of f is $f^n(x,y) = (x+n\alpha, y+\sum_{j=0}^{n-1} g(x+j\alpha))$; if g is smooth, f is a diffeomorphism, and the differential of f^n is

$$Df^n_{(x,y)} = \begin{pmatrix} 1 & 0 \\ \sum_{j=0}^{j=n-1} g'(x+j\alpha) & 1 \end{pmatrix}. \tag{7.3}$$

7.3.2. *Minimality and equicontinuity.*

Proposition 7.9 (Furstenberg)**.** *The homeomorphism f is not minimal if and only if it is conjugate to $(x,y) \mapsto (x+\alpha, y)$ by a homeomorphism $(x,y) \mapsto (x, y+h(x))$ with $h\colon \mathbf{R}/Z \to \mathbf{R}$ that solves the equation $h(x+\alpha) - h(x) = g(x)$.*

This follows from [15]. Indeed, Furstenberg proves that a proper minimal subset of the torus is the graph of such a homeomorphism h.

Proposition 7.10. *If $(f^k)_{k\in\mathbf{Z}}$ is an equicontinuous family, then*

1. *$|\sum_{j=0}^{n-1} g(x+j\alpha)| \leq B$ for some $B > 0$ and all $n \geq 0$;*
2. *g is a coboundary: there is a continuous function $h\colon \mathbf{R}/\mathbf{Z} \to \mathbf{R}$ such that $h(x+\alpha) - h(x) = g(x)$ for all $x \in \mathbf{R}/\mathbf{Z}$;*
3. *f is conjugate to $(x,y) \mapsto (x+\alpha, y)$ by a homeomorphism $(x,y) \mapsto (x, y+h(x))$;*
4. *f is not minimal.*

This is well known to specialists, but we sketch the proof for completeness. The family $f^{\mathbf{Z}}$ is equicontinuous; thus for any $\varepsilon > 0$ one can find $\eta > 0$ such that $dist(f(x,y), f(x',y')) \leq \varepsilon$ as soon as $dist((x,y),(x',y')) \leq \eta$, where $dist$ is the euclidean distance on $\mathbb{T}^2$. Taking ε small, and covering $\mathbb{T} \times \{0\}$ by η^{-1} segments of length $\leq \eta$, one sees that the image of $I \times \{0\}$ by any iterate f^m of f is a curve of length at most ε/η. Now, take $B > \varepsilon/\eta$. Assume that there exists n and x with $|\sum_{j=0}^{n-1} g(x+j\alpha)| > B$. Since the mean of g is 0, the Birkhoff sum $\sum_{j=0}^{n-1} g(x+j\alpha)$ vanish, and one can find an interval $I = [a,b] \subset \mathbb{T}$ such that

$\sum_{j=0}^{n-1} g(a+j\alpha) = 0$, $\sum_{j=0}^{n-1} g(x+j\alpha) > 0$ (or < 0) on $]a,b]$ and $\sum_{j=0}^{n-1} g(b+j\alpha) = B$ (or $-B$). This implies that the segment $I \times \{0\}$ is mapped to a curve of length $\geq B$ by f^n, contradicting the choice of B. This proves Assertion (1).

Because the sums $\sum_0^{n-1} g(x+j\alpha)$ are uniformly bounded, and $x \mapsto x+\alpha$ is a minimal homeomorphism of $\mathbb{T}$, the lemma of Gottschalk and Hedlund (see [20] page 100) shows that there exists a continuous function $h\colon \mathbb{T} \to \mathbf{R}$ satisfying $h(x+\alpha) - h(x) = g(x)$. This proves the second assertion, and the other two follow from it.

7.3.3. *Estimate of the derivatives.* Now, we choose α and g explicitly. We will write g as a Fourier series

$$g(x) = \sum_{k\in\mathbf{Z}} a_k e^{2i\pi kx}. \tag{7.4}$$

Fix a real number $r > 1$. If $a_k \leq r^{-k}$ then g is an analytic function on the circle $\mathbb{T}$. To solve the equation $h(x+\alpha) - h(x) = g(x)$, we also expand h as a Fourier series $\sum_k b_k e^{2i\pi kx}$; then, the b_k must verify $b_k = (e^{2i\pi k\alpha} - 1)^{-1} a_k$ for all $k \neq 0$. Choose

$$\alpha = \sum_{i\geq 1} 10^{-q_i} = 0.100010000000001000...$$

where $q_1 = 1$, and the gaps $q_{n+1} - q_n$ between two consecutive 1s increase quickly; more precisely, we shall assume that

$$q_{n+1} - q_n > (\log(r)/\log(10))10^{q_n}. \tag{7.5}$$

Then, $10^{q_n}\alpha \simeq 10^{-(q_{n+1}-q_n)} \mod 1$. This done, we choose $a_k = 0$ for all indices except the one of the form $k = 10^{q_n}$, in which case we choose $a_k = 10^{-(q_{n+1}-q_n)}$. From Equation (7.5) we get $|a_k| \leq r^{-k}$ for all k, so that g is analytic; but the solutions of the cohomological equation satisfy $b_k = 1$ for $k \neq 1$, and we deduce that there is no L^2 solution h to the cohomological equation. This proves the first assertions of the following proposition.

Proposition 7.11. *There is a pair (α, g) such that α is a Liouville number, g is an analytic function, the cohomological equation $h(x+\alpha) - h(x) = g(x)$ ($\forall x \in \mathbb{T}$) has no continuous solution (resp. no L^2 solution), and the diffeomorphism f satisfies $\| Df^n \|_{\mathbb{T}^2} = o(n^{\varepsilon})$ for all $\varepsilon > 0$.*

Now, we want to find such a pair (α, g) satisfying

$$\sum_{j=0}^{n-1} g'(x+j\alpha) = o(n^{\varepsilon})$$

for every $\varepsilon > 0$. To study this property, we expand g in a Fourier series as in Equation (7.4). Fixing n, we set

$$D_n := \sum_{j=0}^{n-1} g'(x + j\alpha) = 2i\pi \sum_{k\in\mathbf{Z}} \sum_{j=0}^{n-1} \left(e^{2i\pi k\alpha}\right)^j k a_k e^{2i\pi kx}$$

and observe that

$$|\sum_{j=0}^{n-1} \left(e^{2i\pi k\alpha}\right)^j | \leq n \quad \text{and} \quad \sum_{j=0}^{n-1} \left(e^{2i\pi k\alpha}\right)^j = \frac{e^{2i\pi k\alpha n} - 1}{e^{2i\pi k\alpha} - 1}$$

for all $n \geq 1$. Once ε has been fixed, we set $\tau = \varepsilon/4$ and split the sum D_n in two parts:

$$|D_n| \leq 2\pi \left| \sum_{|k|\leq n^\tau} \left(e^{2i\pi k\alpha n} - 1\right) \frac{k a_k}{e^{2i\pi k\alpha} - 1} \right| + 2\pi \left| \sum_{|k|\geq n^\tau} n k a_k \right|.$$

Since g is analytic, its derivative is also analytic, and $k a_k \leq C R^{-k}$ for some constants $C, R > 1$. We shall assume that

$$\left| \frac{a_k}{e^{2i\pi k\alpha} - 1} \right| \leq 1,$$

an inequality which is satisfied in the above construction of the pair (α, g). Altogether we get

$$\begin{aligned} |D_n| &\leq 2\pi \times (2n^\tau \times 2n^\tau) + 2\pi \times 2 \times \sum_{k\geq n^\tau} nCR^{-k} \\ &\leq 8\pi \times n^{2\tau} + 4\pi \times n \times C\frac{R}{R-1}R^{-n\tau} \\ &\leq C' n^{2\tau} \end{aligned}$$

because $nR^{-n\tau} \leq n^{2\tau}$ for n large enough. Thus, $D_n \leq C' n^{\varepsilon/2} = o(n^\varepsilon)$, as required.

7.3.4. *Conclusion.* The proof of Theorem 7.5 is now a direct consequence of Propositions 7.9, 7.10 and 7.11, and Lemma 7.3.

Question 7.1. For a minimal skew product on the 2-torus defined by (7.2), what is its polynomial entropy depending on the value of α and the behavior of g ?

8. SLOW GROWTH AUTOMORPHISMS

Theorem 8.1. *Let f be an automorphism of a smooth complex projective variety X. If the derivatives of f satisfy*

$$\max_{0\leq j\leq n} \| Df^n \| = o(n)$$

then f is an isometry of X for some Kähler metric.

The assumption implies that the growth of $(f^*)^n$ on $H^2(X;\mathbf{R})$ is $o(n^2)$, and this implies that $(f^*)^n$ is bounded on $H^{1,1}(X;\mathbf{R})$ because f^* preserves the Kähler cone. Thus, some positive iterate of f is contained in $\mathsf{Aut}(X)^0$. For simplicity, we can therefore assume that $f \in \mathsf{Aut}(X)^0$.

Consider the Zariski closure A of $f^{\mathbf{Z}}$ in $\mathsf{Aut}(X)^0$; again, changing f in another positive iterate, we can suppose that A is a connected algebraic subgroup of $\mathsf{Aut}(X)^0$. Let r be the complex dimension of A. As a topological group, A is isomorphic to $\mathbf{R}^p/\mathbf{Z}^p \times \mathbf{R}^q$ for some pair of integers (p,q) with $p+q = 2r$, and f corresponds to an element (u,v) with $u \in \mathbf{R}^p/\mathbf{Z}^p$ and $v \in \mathbf{R}^q$. If v were 0, then f would be contained in a compact subgroup of $\mathsf{Aut}(X)^0$, and averaging any Kähler form with respect to the Haar measure on this group, we would get an f-invariant Kähler metric. Thus, we may assume that $v \neq 0$. Thus, if S is a Zariski closed subgroup of A that does not contain f, then f^n goes to infinity in A/S as n goes to $+\infty$.

In particular, the general orbit $A(x) = A/S_x$, with $S_x = Stab(x;A)$, is not closed. Such an orbit is a constructible subset of X, and the sequence $f^n(x)$ goes to its boundary as n goes to $+\infty$. But then x would not be a recurrent point for f, and the polynomial entropy of f would be at least 1. We get a contradiction with the estimate on $\| Df^n \|$.

Remark 8.2. The assumption in Theorem 8.1 is global. In some situation, it is sufficient to study the iterates f^n on a large Zariski open. For instance: *if (f^n) is (locally) equicontinous on the complement of a finite set, then either f is contained in a compact subgroup of* $\mathsf{Aut}(X)$, *or X is the Riemann sphere* $\mathbb{P}^1(\mathbf{C})$ *and f is a homography*; *if f is (locally) equicontinuous on the complement of a Zariski closed subset of co-dimension > 1, then some positive iterate of f is in* $\mathsf{Aut}(X)^0$.

Appendix.– Joint work with Junyi Xie

9. APPENDIX I.– ACTIONS OF FREE GROUPS

Recall that an action of a group Γ on a set X is **free** if the stabilizer of any point $x \in X$ is reduced to the trivial subgroup $\{\mathsf{Id}\} \subset \Gamma$.

Theorem 9.1. *Let M be a compact kähler manifold of dimension ≤ 3. If* $\mathsf{Aut}(M)$ *contains a non-amenable subgroup Γ acting freely on M, then* $\dim(M) = 3$ *and M is a compact torus $\mathbf{C}^3/\Lambda$.*

Moreover, there are examples of non-abelian free groups acting freely on some tori of dimension 3. First, we look at low dimensional tori, then we construct such examples, and we conclude with a proof of Theorem 9.1

9.1. **Tori.** Let $A = \mathbf{C}^n/\Lambda$ be a compact torus of dimension n. Every automorphism $f\colon A \to A$ comes from an affine transformation

$$\hat{f}(z) = L(f)(z) + T(f) \tag{9.1}$$

of $\mathbf{C}^n$, where the translation part is a vector $T(f) \in \mathbf{C}^n$ and the linear part $L(f) \in \mathsf{GL}_n(\mathbf{C})$ preserves the lattice Λ. This defines a homomorphism $\mathsf{Aut}(A) \to \mathsf{GL}_n(\mathbf{C})$, $f \mapsto L(f)$. The following assertions are equivalent:

(i) f has no fixed point;
(ii) the image of the linear transformation $(L - \mathsf{Id})$ does not intersect the set $T + \Lambda$.

In particular, if Γ acts freely on A, then $\det(L(f) - \mathsf{Id}) = 0$ for every f in Γ.

Lemma 9.2. *Let A be a complex torus of dimension ≤ 2. If $\Gamma \subset$* $\mathsf{Aut}(A)$ *acts freely on A, then Γ is solvable. In particular, every free subgroup of* $\mathsf{Aut}(A)$ *acting freely on A is cyclic.*

Proof. If $\dim(A) = 1$, then $\mathsf{Aut}(A)$ is solvable. Assume $\dim(A) = 2$, and write $A = \mathbf{C}^2/\Lambda$. A subgroup G of $\mathsf{GL}_2(\mathbf{C})$ such that $\det(L - \mathsf{Id}) = 0$ for every $L \in G$ is solvable; hence, the equivalence of (i) and (ii) implies: if Γ acts freely on A, the groups $L(\Gamma)$ and Γ are solvable. □

9.2. **Examples.**

9.2.1. *Closed, real analytic manifolds.* The group $\mathsf{SO}_3(\mathbf{R})$ contains a non-abelian free group $\Gamma \subset \mathsf{SO}_3(\mathbf{R})$. This is well known, since the existence of such a group is at the basis of the Banach-Tarsky paradox. Now, the action of Γ on $\mathsf{SO}_3(\mathbf{R})$ by left translations is free, and going to the universal cover SU_2 of $\mathsf{SO}_3(\mathbf{R})$, we obtain a free action of a non-abelian free group on a simply connected manifold. These actions being real analytic, the first assertion of the following theorem is proved.

Theorem 9.3. *There are real analytic, free actions of non-abelian free groups on the following real analytic manifolds:*

(1) *the simply connected, compact Lie group* SU_2*;*
(2) *the torus* $\mathbf{R}^3/\mathbf{Z}^3$.

To get the second assertion, consider the lattice $\mathbf{Z}^3$, together with the standard quadratic form of signature $(1,2)$: $Q(x,y,z) = x^2 - y^2 - z^2$. Its group of isometries $\mathsf{SO}_{1,2}(\mathbf{Z})$ is a lattice in the Lie group $\mathsf{SO}_{1,2}(\mathbf{R})$.

Lemma 9.4. *The group* $\mathsf{SO}_{1,2}(\mathbf{Z})$ *contains a free subgroup* Γ *of Schottky type: the eigenvalues of every element* $g \in \Gamma \setminus \{\mathsf{Id}\}$ *form a triple of real numbers* $\lambda_g > 1 > \lambda_g^{-1}$.

Proof. Consider the subgroup G_0 of $\mathsf{SO}_{1,2}(\mathbf{R})$ preserving each connected component of $\{Q(x,y,z) = 1\}$. If g is an element of this group, and g has an eigenvalue of modulus > 1, then its eigenvalues are $\lambda_g > 1 > \lambda_g^{-1}$ for some $\lambda_g > 1$; equivalently, g is a loxodromic isometry of the hyperbolic space $\mathbb{H} = \{(x,y,z) |\, Q(x,y,z) = 1 \text{ and } x > 0\}$. Now, take two loxodromic isometries f and g in $G_0 \cap \mathsf{SO}_{1,2}(\mathbf{Z})$ to which the tennis-table lemma of Fricke and Klein applies (see [9]). Then, the group Γ generated by f and g is a Schottky group. □

Choose such a free group Γ, of rank 2, and fix a pair (a,b) of elements generating Γ. In [13], Drumm and Goldman find a non-empty open subset U of $\mathbf{R}^3 \times \mathbf{R}^3$ such that for every $(s,t) \in \mathsf{U}$, the affine transformations

$$A_s(x,y,z) = a(x,y,z) + s, \quad B_t(x,y,z) = b(x,y,z) + t \tag{9.2}$$

generate a free group acting freely and properly on $\mathbf{R}^3$. The group generated by A_s and B_t is an affine deformation $\Gamma_{s,t}$ of Γ; given any reduced word w in a, b, and their inverses, we get an element $w(A_s,B_t)$ in $\Gamma_{s,t}$, which we can write

$$w(A_s,B_t)(x,y,z) = w(a,b)(x,y,z) + L_w(a,b)(s) + R_w(a,b)(t) \tag{9.3}$$

where $L_w(a,b)$ and $R_w(a,b)$ are elements of the algebra generated by a and b in $\mathsf{End}\,(\mathbf{R}^3)$.

Since a and b are in $\mathsf{SL}_3(\mathbf{Z})$ the group $\Gamma_{s,t}$ acts on the torus $\mathbf{R}^3/\mathbf{Z}^3$. This action is free if, and only if, given any non-trivial reduced word w, and any element (p,q,r) of the lattice $\mathbf{Z}^3$, the equation

$$(\mathsf{Id}-w(A,B))(x,y,z)+(p,q,r)=L_w(a,b)(s)+R_w(a,b)(t) \tag{9.4}$$

has no solution $(x,y,z)\in\mathbf{R}^3$. Fix such a pair $(w,(p,q,r))$. The question becomes: is the vector $L_w(a,b)(s)+R_w(a,b)(t)$ contained in the affine plane $(\mathsf{Id}-w(A,B))(\mathbf{R}^3)+(p,q,r)$? We distinguish two cases. If $(\mathsf{Id}-w(A,B))(\mathbf{R}^3)+(p,q,r)$ is actually a vector subspace, in which case this subspace coincides with $(\mathsf{Id}-w(A,B))(\mathbf{R}^3)$, then we know from Drumm-Goldman result that there is a pair (s,t) for which Equation (9.4) has no solution. If $(\mathsf{Id}-w(A,B))(\mathbf{R}^3)+(p,q,r)$ does not contain the origin, then there is no solution to Equation (9.4) if (s,t) is small enough. Thus, the set $W(w,(p,q,r))$ of parameters $(s,t)\in\mathbf{R}^3\times\mathbf{R}^3$ such that Equation (9.4) has no solution is non-empty; hence, $W(w,(p,q,r))$ is open and dense (as the complement of a proper affine subspace of $\mathbf{R}^3\times\mathbf{R}^3$). By Baire theorem, the intersection of all those open dense subsets is non-empty: this precisely means that there are pairs (s,t) such that the free group $\Gamma_{s,t}$ acts freely on $\mathbf{R}^3/\mathbf{Z}^3$.

9.2.2. *Abelian threefolds.* Consider any lattice $\Lambda_0\subset\mathbf{C}$, for instance the lattice $\Lambda_0=\mathbf{Z}[\mathrm{i}]$. Set

$$\Lambda=\Lambda_0\times\Lambda_0\times\Lambda_0\subset\mathbf{C}^3 \tag{9.5}$$

and denote by N the abelian threefold $\mathbf{C}^3/\Lambda$. Now, copy the last argument, with the same group $\Gamma_{s,t}$, but viewed as a subgroup of the affine group $\mathsf{SL}_3(\mathbf{Z})\ltimes\mathbf{C}^3$, acting on $N=\mathbf{C}^3$. We get

Theorem 9.5. *Let $\Lambda_0\subset\mathbf{C}$ be a cocompact lattice. There is a free action of a non-abelian free group on the abelian threefold $(\mathbf{C}/\Lambda_0)^3$ by holomorphic affine transformations.*

9.3. **Proof of Theorem 9.1.** We now prove Theorem 9.1. According to [], the group $\mathsf{Aut}(M)$ satisfies Tits alternative: *if $\Gamma\subset\mathsf{Aut}(M)$ does not contain a non-abelian free group, then Γ contains a solvable subgroup of finite index and, in particular, is amenable.* Thus, we can, and shall, assume that Γ is a non-abelian free group, acting freely on some compact kähler manifold M of dimension ≤ 3. We shall use several times the following fact.

Lemma 9.6. *If the group Γ stabilizes a subset $S\subset M$, the restriction $f\in\Gamma\to f_{|S}$ is an injective morphism, and the action of Γ on S is free. If the action of a* *finite index subgroup* $\Gamma_0 \subset \Gamma$ *lifts to an action on a finite cover* $M' \to M$*, then the action of* Γ_0 *on* M' *is free.*

9.3.1. *Kodaira dimension.*

Lemma 9.7. *Let* M *be a compact kähler manifold, and let* Γ *be a subgroup of* $\mathsf{Aut}(M)$*. If the kodaira dimension of* M *is non-negative, either* K_m *is torsion, or there is a finite index subgroup* Γ_1 *of* Γ *and a* Γ_1*-invariant, proper, non-empty, and irreducible Zariski closed subset* $Z \subset M$*.*

Proof. If the kodaira dimension of M is non-negative, the Kodaira-Iitaka fibration provides a surjective morphism $\pi \colon M \to B$ such that:

(1) π is $\mathsf{Aut}(M)$-equivariant: there is a homomorphism $\rho \colon f \in \mathsf{Aut}(M) \mapsto f_B \in \mathsf{Aut}(B)$ such that $f_B \circ \pi = \pi \circ f$;
(2) the image $\rho(\mathsf{Aut}(M)) \subset \mathsf{Aut}(B)$ is a finite group. (see [37]).

Thus, a finite index subgroup of Γ fixes individually every fiber of π. If $\dim(B) \geq 1$, we take Z to be an irreducible component of some fiber, and Γ_1 the finite index subgroup of Γ that preserves the fiber, as well as every irreducible component of this fiber. If $\dim(B) = 0$, the kodaira dimension of M is 0, and we can fix an integer $d > 0$ such that $H^0(M, K_M^{\otimes d}) = \mathbf{C}\Omega$ for some non-trivial section Ω of $K_M^{\otimes d}$. There is a homomorphism $\xi \colon \Gamma \to \mathbf{C}^*$ such that

$$f^*\Omega = \xi(f)\Omega \tag{9.6}$$

for every $f \in \Gamma$. In particular, the divisor $(\Omega)_0$ is Γ-invariant. If this divisor is empty, then $K_m^{\otimes d}$ is the trivial bundle. If not, we define Z to be an irreducible component of $(\Omega)_0$. □

9.3.2. *Curves and surfaces.*

Lemma 9.8. *If a free group acts freely on a curve, the group is cyclic. If* $\dim(M) \leq 2$ *and* Γ *is a free group acting freely on* M*, then* Γ *does not stabilize any proper, non-empty, Zariski closed set.*

Proof. Since every automorphism of $\mathbb{P}^1(\mathbf{C})$ has a fixed point, we can assume that the genus of the curve is at least 1, but then its automorphism group is virtually solvable, and any free subgroup is cyclic. The second assertion follows from Lemma 9.6. □

Lemma 9.9. *If a free group acts freely on a compact, kähler surface, the group is cyclic.*

Proof. If $h^{2,0}(M) > 0$, Lemmas 9.7 and 9.8 imply that K_M is trivial. If M is a K3 surface, its Euler characteristic is positive, every homeomorphism of M has a periodic point, and we get a contradiction. If M is a torus, we get a contradiction from Lemma 9.2. This exhausts all surfaces with K_M trivial.

Now, assume that $h^{2,0}(M) = 0$. Then, from the holomorphic Lefschetz fixed point formula, we must have $h^{1,0}(M) > 0$. Consider the Albanese morphism $\alpha\colon M \to A_M$, where A_M is the Albanese torus of M, and let E be the image of α. Since $h^{1,0}(M) > 0$, we get $\dim(E) \in \{1,2\}$. By Lemma 9.6 and 9.8, every proper Γ-invariant analytic subset of M (resp. of E) is empty. Thus, E is smooth and α is a submersion. First, assume $\dim(E) = 1$. Since $E \subset A_M$ can not be the Riemann sphere, its group of automorphisms is virtually solvable; thus, the kernel of the homomorphism $\Gamma \to \mathsf{Aut}(E)$ is a non-abelian free group, acting freely on the fibers of α, contradicting Lemma 9.8. Thus, $\dim(E) = 2$ and $\alpha\colon M \to E$ is a finite cover. This implies $h^{2,0}(M) > 0$, and we get a contradiction. □

9.3.3. *Dimension* 3. Assume $\dim(M) = 3$, M is not a torus, and the non-abelian free group Γ acts freely on M.

a. kod$(M) \geq 0$. – First, assume that the kodaira dimension of M is non-negative. It follows from Lemmas 9.7 to 9.9 that K_M is torsion. Then, after a finite étale cover, K_M is trivial and M is one of the following examples:

(1) a torus of dimension 3;
(2) a (simply connected) Calabi-Yau threefold;
(3) a product of an elliptic curve with a K3 surface.

Lemma 9.10. *If a finite cover of M is a torus, and $\Gamma \subset \mathsf{Aut}(M)$ is a non-abelian free group acting freely on M, then M is a torus.*

Thus, the first case is excluded, since we assume that M is not a torus.

Proof. By assumption, there is a torus $A = \mathbf{C}^3/\Lambda$, and a finite group G acting freely on A such that $M = A/G$. By construction, M is a quotient of $\mathbf{C}^3$ by a group of affine transformations $\tilde{G} \subset \mathsf{Aff}(\mathbf{C}^3)$; the group Λ is a finite index subgroup of $\tilde{G}$, and the image of the (linear part) homomorphism $L\colon \tilde{G} \to \mathsf{GL}_3(\mathbf{C})$ is a finite subgroup (isomorphic to G since the action of G on A is free).

The group Γ lifts to a free group of affine transformations of $\mathbf{C}^3$ permuting the orbits of $\tilde{G}$. When f is an element of Γ, we denote by $\hat{f}\colon z \mapsto L(f)z + T(f)$

the corresponding affine transformation. The group $L(\Gamma)$ normalizes $L(\tilde{G})$, and a finite index subgroup $L(\Gamma_0)$ commutes to every element in $L(\tilde{G})$. If G is non-trivial, then $L(\tilde{G})$ contains a non-trivial linear transformation S, S is diagonalizable (because it has finite order), and $L(\Gamma_0)$ preserves its eigenspaces. Since the action of G on A is fixed-point free, the eigenspace E_1 corresponding to the eigenvalue 1 has positive dimension, intersects Λ on a lattice $\Lambda_1 = E_1 \cap \Lambda$, and both E_1 and Λ_1 are invariant under the action of $L(\Gamma_0)$. If S had three distinct eigenvalues, the three eigenlines of S would be $L(\Gamma_0)$ invariant, contradicting the fact that Γ is not virtually solvable. Thus, S has exactly one other eigenvalues α, corresponding to a $L(\Gamma_0)$-invariant eigenspace E_α, with $E_1 \oplus E_\alpha = \mathbf{C}^3$. Assume $\dim(E_1) = 1$ and $\dim(E_2) = 2$. Then, a free subgroup $L(\Gamma_1)$ acts trivially on E_1 (because $\mathsf{GL}(E_1)$ is abelian), and its action on E_2 is made of matrices with eigenvalues $\neq 1$. Take a generating pair f, $g \in \Gamma_1$; computing the commutator $[\hat{f}, \hat{g}]$, we observe that the translation part $T([f,g])$ is contained in E_α and that $[\hat{f}, \hat{g}]$ has a fixed point in $\mathbf{C}^3$. This contradicts the assumption on Γ. The case $\dim(E_1) = 2$ is similar. Thus, $L(\tilde{G})$ is trivial and M is actually a torus. □

Assume we are in case (3), with a finite cover M' of M isomorphic to $X \times E$ for some $K3$ surface X; there is a finite group of automorphisms $F \subset \mathsf{Aut}(M') = \mathsf{Aut}(X \times E)$ acting freely on M' such that $M = M'/F$. Every automorphism f of M' preserves the Albanese fibration $\alpha' \colon M' \to E$: this gives a homomorphism $F \to \mathsf{Aut}(E)$, and we denote by F_E its image. The fibration α' determines a fibration $\alpha \colon M \to E/F_E$, and this fibration is Γ invariant. Two cases may occur. Either E/F_E is a curve of genus 1, its automorphism group is solvable, and we get a contradiction with Lemma 9.9. Or E/F_E is a Riemann sphere, the fixed points of the elements of F_E correspond to the critical values of α, and a finite index subgroup of Γ preserves the corresponding fibers. Again, Lemma 9.9 provides a contradiction.

We can now assume that we are in case (2), i.e. the universal cover of M is an irreducible Calabi-Yau threefold. Lifting Γ to the universal cover, we can assume that M itself is Calabi-Yau. The action on cohomology gives rise to a homomorphism $\Gamma \to \mathsf{GL}(H^*(M;\mathbf{Z}))$, $f \mapsto f_* = (f^{-1})^*$. Here is the key lemma:

Lemma 9.11. *Let M be a (simply connected) Calabi-Yau threefold. The action of $\mathsf{Aut}(M)$ on the cohomology group $H^3(M;\mathbf{Z})$ factors through a finite group.*

Let f be an automorphism of M. If all orbits of f are infinite then

(1) *the action of f on the cohomology of M is virtually unipotent: there is a positive integer k such that $(f^k)^*$ is unipotent;*
(2) $h^{2,1}(M) = h^{1,1}(M)$ *and the topological Euler characteristic of M is* 0.

Proof. Fix a kähler class κ on M. Since $h^{p,q}(M) = 0$ when $p+q = 1$ or 5, we deduce that every class α in $H^{2,1}(M)$ is primitive: $\alpha \wedge \kappa = 0$ because $H^{3,2}(M) = 0$. Thus, the intersection product $\int_M \alpha \wedge \overline{\alpha}$ determines an $\mathsf{Aut}(X)$-invariant, positive definite quadratic form on the vector space $H^{2,1}(M)$. On $H^{3,0}(M)$, the product $\omega \mapsto \int_M \omega \wedge \overline{\omega}$ is also positive definite. As a consequence, the image of $\mathsf{Aut}(M)$ in $H^3(M;\mathbf{C})$ is contained in a unitary group. Since it preserves the integral structure $H^3(M;\mathbf{Z})$, it is contained in a finite group. This implies that a finite index subgroup of $\mathsf{Aut}(M)$ acts trivially on $H^3(M,\mathbf{Z})$.

Now, apply the holomorphic Lefschetz fixed point theorem: if there is no periodic point, the traces of f^* on $H^{1,1}(M)$ and $H^{2,1}(M)$ satisfy

$$Tr(((f^n)^*)_{1,1}) = Tr(((f^n)^*)_{2,1}) \quad (\forall n \in \mathbf{Z}\setminus\{0\}). \tag{9.7}$$

Changing f in a positive iterate $g = f^m$, we may assume that $Tr(((g^n)^*)_{2,1}) = h^{2,1}(M)$ for all n, and then $Tr(((g^n)^*)_{1,1}) = h^{2,1}(M)$ also for all n. But this equality implies that $(g^*)_{1,1}$ is unipotent and $h^{2,1}(M) = h^{1,1}(M)$. Thus, $(f^*)_{1,1}$ is virtually unipotent. Since $H^{2,0}(M) = 0$, the action of f^* on $H^*(M,\mathbf{Z})$ is virtually unipotent. □

Since Γ is a free subgroup of $\mathsf{Aut}(M)$, the representation $\Gamma \to \mathsf{GL}(H^*(M;\mathbf{Z}))$ is faithfull because the kernel of $\mathsf{Aut}(M) \to \mathsf{GL}(H^*(M;\mathbf{Z}))$ is finite when M is Calabi-Yau (every holomorphic vector field on M is 0). Since Γ acts freely on M, all elements $f^* \in \mathsf{GL}(H^*(M;\mathbf{Z}))$ for $f \in \Gamma$ are virtually unipotent: this implies that Γ is cyclic, because a subgroup of $\mathsf{GL}_m(\mathbf{C})$ all of whose elements are virtually unipotent is a solvable group up to finite index. We obtain the following.

Lemma 9.12. *Let M be a compact kähler manifold of dimension* 3 *whose kodaira dimension is non-negative. If there is a non-abelian free group acting freely on M, then M is a torus.*

b. kod$(M) = -\infty$. – Thus, in what follows, we assume that the Kodaira dimension of M is $-\infty$. In particular, $h^{3,0}(M) = 0$. We prove that non-abelian free groups can not act freely on M.

Assume that $h^{1,0}(M) = 0$, and apply the holomorphic fixed point formula. This gives

$$Tr((f^n)^*_{2,0}) = -1 \tag{9.8}$$

for all elements $f \neq \mathsf{Id}$ of Γ and all integers $n \neq 0$. Again, we get a contradiction. Thus, $h^{1,0}(M) > 0$, and the Albanese map is a non-trivial morphism

$$\alpha\colon M \to A_M \tag{9.9}$$

where A_M is the Albanese torus of M. This map is equivariant with respect to a homomorphism $\rho\colon \mathsf{Aut}(M) \to \mathsf{Aut}(A_M)$, meaning that $\alpha \circ f = \rho(f) \circ \alpha$ for every $f \in \mathsf{Aut}(M)$. Let $E \subset A_M$ be the image of α. Assuming that the rank of the free group Γ is at least 2, we shall prove successively that

- E is smooth and the map $\alpha\colon M \to E$ is a submersion;
- E has dimension 2.

If E contains a non-empty Zariski closed proper subset Z which is invariant under the action of $\rho(\Gamma)$, then $\alpha^{-1}(Z)$ is a non-empty, Zariski closed, proper and Γ-invariant subset of M; its dimension is at most 2, and we get a contradiction since Γ can not act freely on such a subset (Lemmas 9.6, and 9.9). Thus, E is smooth and the critical locus of α is empty, i.e. α is a submersion.

If E is a curve, its genus is ≥ 1 (because E is contained in the torus A_M), its automorphism group is solvable, and there is a non-abelian free group $\Gamma_1 \subset \Gamma$ that acts trivially on E, and freely on every fiber of α: again, we get a contradiction.

If $\dim(E) = 3$, then M is a finite cover of E (because α is a submersion). This implies that there is a non-trivial holomorphic 3-form on M, contradicting $h^{3,0}(M) = 0$.

Thus, we assume $\dim(E) = 2$. Let $K \subset A_M$ be the (connected) subtorus of maximal dimension such that $E + K = E$: it is uniquely determined by E, and the projection $p(E)$ of E in the quotient torus A_M/K is a manifold of general type and of dimension $\dim(E) - \dim(K)$ (more precisely, the canonical bundle of $p(E)$ is ample, see [10], §VII). If $K = \{0\}$ is reduced to a point, then, E is a surface of general type, hence $\mathsf{Aut}(E)$ is a finite group and we get a contradiction with Lemma 9.8. If $\dim(K) = 1$ we get $\dim(p(E)) = 1$ and the morphism $p \circ \alpha\colon M \to p(E)$ is invariant under a finite index subgroup of Γ. Since $p(E)$ is a curve of general type, the image of Γ in $\mathsf{Aut}(p(E))$ is finite and, again, we get a contradiction with Lemma 9.9.

Now, assume $\dim(K) = 2$, which means that $E = A_M$ is a 2-dimensional torus. Denote by (x, y) the affine coordinates on $E = \mathbf{C}^2/\Lambda$, where Λ is a lattice in $\mathbf{C}^2$. Let Ω be a holomorphic 2-form on M. Since $h^{3,0}(M) = 0$, we get

$$\Omega \wedge \alpha^*(dx) = \Omega \wedge \alpha^*(dy) = 0. \tag{9.10}$$

This implies that $\Omega = a\alpha^*(dx\wedge dy)$ for some holomorphic function $a\colon M\to \mathbf{C}$; such a function must be a constant, and we conclude that $H^{2,0}(M) = \mathbf{C}\alpha^*(dx\wedge dy)$. In particular, a finite index subgroup of Γ acts trivially on $H^{2,0}(M)$, and the holomorphic Lefschetz fixed point formula gives $1 - Tr((f^*_{1,0})^n) + 1 = 0$ for every $f\in\Gamma\setminus\{\mathsf{Id}\}$ and every $n\neq 0$. This shows that $f^*_{1,0}$ is unipotent. Hence, $\rho(\Gamma)$ is a subgroup of $\mathsf{Aut}(E)$, all of whose elements are affine automorphisms of $\mathbf{C}^2/\Lambda$ with a unipotent linear part: such a group is solvable. Thus, a non-abelian free subgroup of Γ acts freely on the fibers of α and this contradicts Lemma 9.8.

This concludes the proof of Theorem 9.1

10. Appendix II.– Theorems of Hermann, and of Lesieutre, Oguiso and Zhang

Michael Herman proved that there is a (real analytic) diffeomorphism h of a (real analytic) compact manifold M such that (1) the topological entropy of $f\colon M\to M$ is positive, and (2) f is a minimal transformation of M, meaning that every orbit of f is dense in M (for the euclidian topology). We don't know whether such an example exists in the context of holomorphic diffeomorphisms of compact kähler manifolds, even for Calabi-Yau manifolds. We explain now that such an example can not exist in dimension ≤ 3.

In [28], Lesieutre studied automorphisms of complex projective manifolds of dimension 3 with positive topological entropy, showing how results of the minimal model program interact with techniques from dynamical systems. This is useful to study automorphisms acting minimally, and Oguiso and Zhang recently announced the following result: let f be an automorphism of a complex projective manifold X of dimension 3; if all orbits of f are Zariski dense, then either X is a torus or a finite cover of X is a (simply connected) Calabi-Yau manifold([1]). From Lemma 9.11, we know that a holomorphic diffeomorphism of a Calabi-Yau threefold with positive entropy has a periodic orbit. Thus, with Oguiso-Zhang theorem we get

Theorem 10.1. *Let f be an automorphism of a complex projective manifold of dimension at most* 3. *If the topological entropy of $f\colon M\to M$ is positive, there is a non-empty, proper, Zariski closed, and f-invariant subset $Z\subset M$; in particular, the action of f on M is not minimal.*

[1]The conjecture is that this last case should be excluded too, so that X is in fact a torus.

This result says that there is no holomorphic Herman example on compact kähler manifolds of dimension ≤ 3. We believe that such an example can be constructed if the kähler assumption is taken out. Mary Rees in [36] constructs Herman-like examples for tori of any dimension (starting from 2) - these examples are only C^0-smooth.

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