NOTES ON TILING BILLIARDS : SOME THOUGHTS AND QUESTIONS

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A tiling billiard is a mathematical model of mouvement of light in the heterogeneous medium. Consider a tiling of the euclidian plane by polygones for which every tile t is marked by a number $n(t) \in \mathbb{R}^*$ which is its refraction index. A billiard in this tiling is constructed in a following way. A particle follows a straight line segment till a moment when it strikes into a boundary of some tile. Then, it passes to a neighbouring tile and the direction of its trajectory changes with respect to Snell-Descartes law. These billiards were firstly mentionned by Mascarenhas and Fluegel [34] in a non-published preprint. The mathematical study of these billiards was proposed in 2016 [20] for the case of a square tiling and in [12] for any tiling where the coefficient of refraction between two tiles is always equal to -1. This second case doesn't (yet?) correspond to any physical reality. But it looks promising: recent progress in meta-materials has showed [41, 45] the existence of metamaterials with negatif refraction index (around -0.6). Note that most of usual plastic or glass materials have coefficients of refraction bigger than 1. The case negative refraction index equal to -1 represents, in our opinion, a great mathematical interest, as it will be explained in these notes.

A complete bibliography on tiling billiards contains around ten articles (see the introductions of [25, 9]). The goal of these notes is to present some thoughts on tiling billiards as well as ask some questions that seem interesting to us, in order to maybe motivate our reader to engage herself in this beautiful world. Thoughts and questions, that's all it is.

1. Locally foldable tilings

In our work with Pascal Hubert [25], we give a complete description of qualitative behavior of tiling billiards in triangular periodic tilings with a coefficient of refraction equal to -1. Here, a **triangular periodic tiling** is a tiling of a plane by three families of equidistant parallel lines. See [12] for the first definition of this system and [9, 25] for its interesting properties. In [25] we also study the case of periodic tilings by cyclic quadrilaterals. A **periodic tiling by cyclic quadrilaterals** is a tiling in which all tiles are isometric to some cyclic (inscribed in a circle) quadrilateral and each two neighbouring tiles are centrally symmetric to each other with respect to the center of their common side.

We think that one can generalize some of the results concerning triangular periodic tilings and tilings by cyclic quadrilaterals to the case of locally foldable tilings. This Section concerns hence this larger class of tilings and tiling billiards (always with coefficient -1 between neighbouring tiles) ans problems that arise from the study of such kind of dynamics.

A tiling of plane by polygons is **locally foldable** if

(a.) every vertex of this tiling has an even degree, i.e. an even number of polygons meets in each of the vertices and such a tiling can be colored in two different colors (say, black and white for the following)

(b.) the sume of adjacent angles of a fixed color in each vertex of a tiling is equal to π (equilibrium of the angles)

Example. Many more periodic locally foldable tilings exist. For example, a Napoleon's tiling obtained by four different types of types : one tile is an arbitrary triangle with sides of lengths a, b, c, and three other types are three different equilateral triangles with the sides of the lengths a, b and c

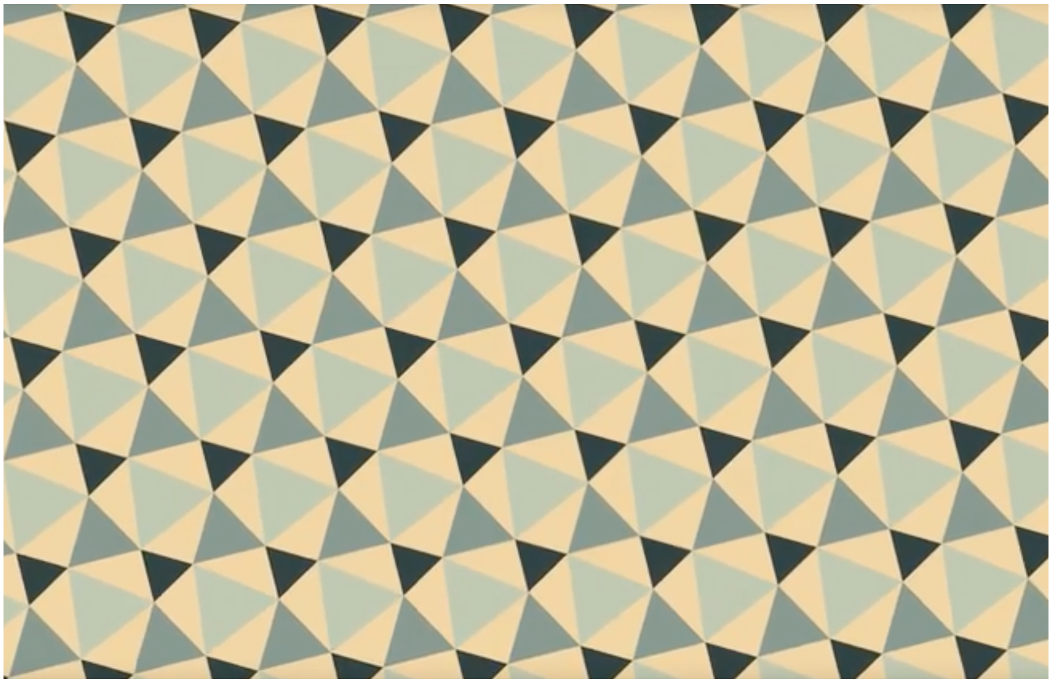

FIGURE 1. Napoleon's tiling, screenshot of a *Think Twice* video

correspondingly. [1] In general, many periodic non-locally foldable tilings can be constructed, see e.g. the relation between periodic locally foldable tilings and dimer models [28], as well as Hull's survey [26].

The interest of the class of locally foldable tilings is that this class admits natural foliations with leafs that are orbits of a tiling billiard (these foliations will be discussed in the following). This helps to prove that such a billiard is a **closed dynamical system**: all of its bounded orbits are closed. This is a new and simple remark, and we think it should be taken into account for the study of these billiards. We write this fact for everybody, ourselves included - author's own work [25] with P. Hubert gives the proofs of some results on triangle tiling billiards without using the fact that we are dealing with a foliation.

For every fixed combinatorics (given by a bipartite graph G), a periodic locally foldable tiling can be parametrized by a finite number of parameters, and one can define the Lebesgue measure on the set of such tilings of fixed combinatorics. We hope that one can use the methods of complex analysis (discrete complex analysis?) as well as of complex reflection in order to study such tilings.

1.1. **Dimers and Harnak curves: parameters of a locally foldable tiling with fixed combinatorics.** It happens that locally foldable tiling appear in a very different context, i.e. in the study of probabilistic dimer model. For example, in [28] the parameters on the space of locally foldable tilings with a given *periodic* bipartite graph structure are defined: a point in a space of parameters defining a fixed tiling has the coordinates defines by so-called X variables and a point (λ_1, λ_2) on a spectral curve, see [28, 29]. If a point on a spectral curve is simple, Lemma 10 in [28] says that the corresponding tiling is periodic. If it is a double point, then the space of parameters corresponding to such a point has 4 degrees of liberty but, following Lemma 11 in [28], only one of them gives a periodic origami.

[1] We learned about the existence of this tiling and its relation to a so-called Napoleon's theorem in the video of a wondeful Youtube channel *Think twice*, follow the link for this video and many other beautiful math videos, `https://www.youtube.com/watch?v=KQ8cSuoopyc`.

Question 1. Is the dynamics of a tiling billiard in a locally foldable tiling given by a (periodic) bipartite graph G almost always "the same"and **integrable**: for a fixed combinatorics (graph G) and a full measure on the set of tilings and initial conditions on the trajectories, the trajectories are closed or escape to infinity in a linear way ?

Any answer is interesting. For the "yes" direction, one could ask oneselves if this question is related to Novikov's problem [38] on the asymptotic behavior of plane sections of three-periodic surfaces. For the "no" direction, if one finds the stable examples of new behaviors on some locally foldable tiling, one can ask oneselves if the generic behavior of trajectories in such a billiard is related to the corresponding Harnack curve.

It is interesting to note that locally foldable tilings have already been studied in many other contexts. Indeed, they are related (as shown in [28]) to circle patterns and three dimensional *hyperbolic* polyhedra.

We find it very interesting to study the connections between the dimers and the tiling billiards. Maybe, by chance, there are some beatiful results between probability, dynamics and combinatorics?... Of course, it is interesting to look at a more general case of non-periodic locally foldable tilings.

1.2. **Tiling billiards in locally foldable tilings.** An important property of locally foldable class of tilings is that the tiling billiards on these tilings have wonderful rigidity properties.

This Lemma is obvious for origamists but we formulate and prove it here since it is a key Lemma for the following.

Lemma 1 (Folding lemma for locally foldable tilings)**.** *Fix a locally foldable tiling (not necessarily periodic) and some tile τ_0 in it. Then take any other tile τ (maybe equal to τ_0) and any path through the tiles of a tiling connecting τ_0 to τ. Then the following holds:*

(a.) *one can fold the neighbouring polygons one on each other in such a way that each two neighbouring polygones in a path are folded along their common edge*
(b.) *the folded image of τ is well defined - doesn't depend on a chosen path.*

Proof. It suffices to prove the statement for $\tau = \tau_0$ (and a path being a loop), and even more, by breaking down each closed path in a sum of paths around vertices, it suffices to prove this lemma for the closed path going around a vertex contained in τ_0. The statement of the lemma for such a loop is equivalent to local foldability condition. Indeed, when one folds one polygon on another in a tour around a vertex, the difference between black and white angles in the vertex defines the displacement of the initial tile τ_0 with respect to its initial position. Since by definition this diffrence is zero in a locally foldable tiling, a tile comes back to its place, see Figure 2. □

One can speak about a holonomy of a tiling, the holonomy of a locally foldable tiling being trivial.

This lemma immediately gives a much simpler proof of the results of Section 2 in [9] and generalizes them. Indeed, we have the following

Lemma 2 (Closed trajectories)**.** *Any trajectory of a tiling billiard with coefficient of refraction equal to -1 in a locally foldable tiling passes through each tile a finite number of times, and all bounded trajectories are close. For any convex tile, a trajectory passes through it at most once. Moreover, each periodic trajectory is stable under perturbation (slight change of the form of each tile as well as slight change of initial conditions).*

Proof. By Lemma 1, the folding of the tiling is well defined. By the simple and crucial remark first noticed in [9] for triangle tiling billiards, for any trajectory, its image in the folded figure is contained in some line. The following is elementary. This line crosses each polygon a finite number of times. For a bounded trajectory, at some moment the trajectory will pass by the same polygon, hence it will repeat itself and close up. Note that for convex polygons, the trajectory crosses them at most once. And in any case, a bounded trajectory corresponds ot a simple closed curve. Moreover, each periodic

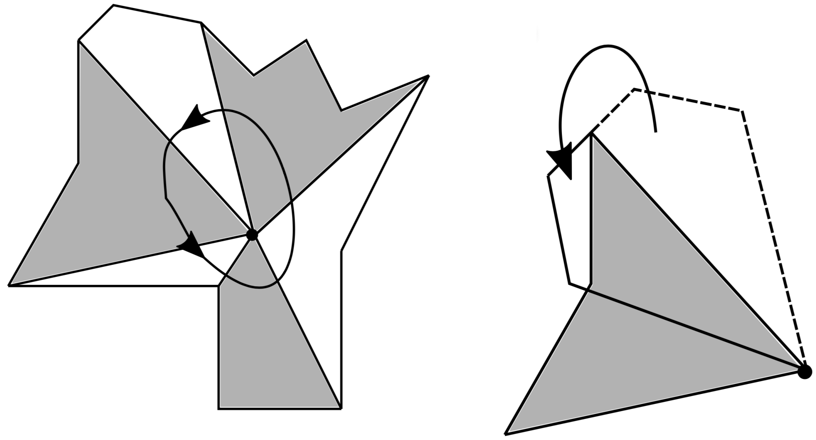

FIGURE 2. *Folding around a vertex.* On the left picture one can see a vertex of a locally foldable tiling and the polygons that contain this vertex. One makes a loop around a vertex and folds each neighbouring polygon on the previous one as shown on the picture on the right. In a locally foldable tiling, the image of an initial polygon under this procedure coincides with its initial position on the plane.

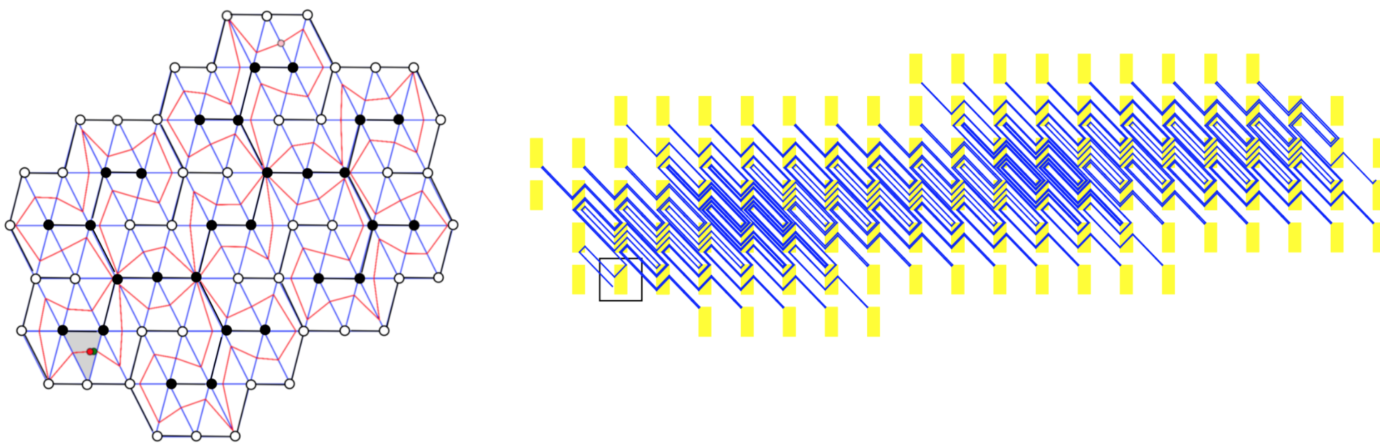

FIGURE 3. On the left : illustration of Tree conjecture in its generalisation, that of density. The trajectory is drawn in red and this is a trajectory in a triangle tiling billiard, with internal (external) graphs with colored in black (white) vertices. On the right : an example of a trajectory in a wind-tree tiling billiard.

trajectory is stable under perturbation since its symbolic dynamics is not changed if the line changes its direction a little bit. □

Remark. The statement of this Lemma holds for any tiling with differentiable by arcs boundaries having a locally foldable condition (that can be generalized far more than for the case of polygons).

2. Tree conjecture and limits of trajectories

In the beginning of this Section, we concentrate our attention on a very small class of tiling billiards, triangle tiling billiards. This class was first studied in [9] and the following conjecture was formulated concerning triangle tiling billiards: every periodic trajectory γ of a triangle tiling billiard doesn't go around the triangles, see Figure 3 for illustration.

Conjecture 1 (Tree conjecture)**.** *Let Λ be as a union of all vertices and edges of all drawn triangles in a periodic triangle tiling. Take any periodic closed trajectory δ of a corresponding triangle tiling billiard. It incloses some bounded domain $U \subset \mathbb{R}^2$ in the plane, $\partial U = \delta$ and $U \cap \Lambda$ is an embedding of some graph in the plane. Then this graph is a* tree.

This conjecture was proven in [9] for obtuse triangles (the tree in question is in this case a chain). This conjecture has an even stronger form corresponding to the fact that any trajectory (not necessarily closed) fills in the subset of the plane that it occupies, that we present in the following. We did advance in the proof of this conjecture but still didn't finish it entirely.

Thanks to the study of foliations of the plan naturally associated to such a tiling billiard, we have succeeded to reduce the Tree conjecture to the conjecture on the local behavior of separatrices of a vertex in a tiling. We believe that the behavior described in the Conjecture 1 is related to certain properties of balance of the symbolic dynamics of fully flipped interval exchange transformations (belonging to the kernel of SAF invariant, see Section 5).

The Tree conjecture has a generalisation that we call Density conjecture. The idea is that every trajectory (not necessarily a closed one) *constructs* by its dynamics two graphs simultaneously: one interior and one exterior. We also find some obstructions to the Tree conjecture on the locally foldable tilings, and give examples when this conjecture doesn't hold anymore.

Our interest to this conjecture (besides its intrinsic beauty) comes from the following remark. If this conjecture is true, it may be a way to prove the existence of the exceptional trajectories of triangle tiling billiards that pass by *all* the triangles without exception. In the preprint [9], the authors claim to have constructed some of such examples but it seems that the proof is not finished. Such exceptional trajectories (if they exist, and we strongly believe so) will be related to the exceptional fractal curves studied by McMullen [36], Arnoux [2], Hooper-Weiss [23], Lowenstein-Poggiaspalla-Vivaldi [31]. These diffrent curves are all connected to each other and are extremely important in order to understand families of interval exchange transformations. It would be extremely interesting to find the trajectories of triangle tiling billiards that approach the Rauzy fractal and prove the convergence results. As announced in [9], these trajectories seem to experimentally exist although we are far away from proving it. In the context of the study of foliations corresponding to Arnoux-Yoccoz interval exchange transformations, the convergence of certain triangle tiling billiard trajectories to the Rauzy fractal has been conjectured by Hooper and Weiss in [23].

2.1. **Generalisation of the Tree conjecture : Density conjecture.** In this Section we give a generalization of the Conjecture 1.

Denote $\mathcal{V}$ a set of vertices $v \in \Lambda$ such that the trajectory intersects at least one edge with v as an extremity. We will color the vertices of this set in two colors, black and white : $\mathcal{V} = \mathcal{B} \sqcup \mathcal{W}$ by following the trajectory. Here is an algorithm of simultaneous construction of the sets $\mathcal{B}$ and $\mathcal{W}$.

First, pick some edge e crossed by a trajectory. Denote its extremities b_0 and w_0, in any order. Add $b_0 \in \mathcal{B}, w_0 \in \mathcal{W}$. Then we continue the procedure by adding, after each step of a tiling billiard reflection, the points b_j, w_j being the extremities of the sides crossed by the trajectory to the collection $\mathcal{B} \sqcup \mathcal{W}$ by assigning them colors. The colors are assigned in such a way that the edges of $b_j b_{j+1}$ and $w_j w_{j+1}$ are not crossed by the trajectory and, on the contrary, the edges $b_j w_{j+1}$ and $w_j b_{j+1}$ are crossed by the trajectory. Note that first, some of these edges degenerate into vertices (at each step, either $b_j = b_{j+1}$ or $w_j = w_{j+1}$). And second, it may also happen that $b_j = b_k, k < j-1$. See Figure 3 for an example.

Take two graphs $\Gamma_{\mathcal{B}}$ and $\Gamma_{\mathcal{W}}$ in the plane with vertices being correspondingly the sets $V_{\mathcal{B}} := \mathcal{B}$ and $V_{\mathcal{W}} := \mathcal{W}$ and the edges connecting two vertices with consequent indices of the same color. If $b_j = b_{j+1}$, we do not add any loop.

Then Conjecture 1 can be generalized to have the following form:

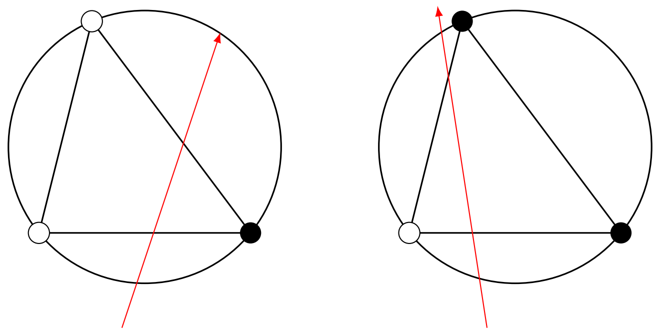

FIGURE 4. The edge that has two different colored vertices is crossed by the trajectory. One can easily see that all the vertices of the same color end up after folding on the same side from the line of trajectory in the circumcircle system.

Conjecture 2 (Density conjecture)**.** *A trajectory is not closed if and only if both of the corresponding graphs* $\Gamma_{\mathcal{B}}$ *and* $\Gamma_{\mathcal{W}}$ *are trees. A trajectory is closed if and only if one of the corresponding graphs is a tree (corresponding to the vertices inside the trajectory) and another of these graphs(corresponding to the vertices outside of the trajectory) has a unique cycle in it.*

To a triangle tiling billiard one associates a system of reflections in a circumcircle (see Definition 3 in [25]). The triangles obtained in a trajectory are exactly those all of whose vertices are colored (belong to $\mathcal{V}$). By coloring the images of black and white vertices in the tiling for the system in a circumcircle (a *folded* system) as well, one can note that black vertices lay on one of side of the chord defined by a trajectory and white vertices lay on the other side.

Proposition 1. *Fix some trajectory of a triangle tiling billiard. Take a parameter* τ *corresponding to the direction of this trajectory and the sets* $\Gamma_{\mathcal{B}}$ *and* $\Gamma_{\mathcal{W}}$ *defined by this trajectory. Then for the corresponding system of reflections in a circle with the same* τ *and a corresponding initial condition, all of the images of the vertices of* $\Gamma_{\mathcal{B}}$ *will be on one side of the chord defined by* τ *and all of the vertices of* $\Gamma_{\mathcal{W}}$ *will be on the other side.*

Proof. This is a very simple geometrical argument by induction that uses the fact that the triangle tiling folds into the circle, see Figure 4. □

This remark helps to prove that a trajectory can't contour a triangle without getting inside it.

Proposition 2. *There is no trajectory of triangle tiling billiard not passing through some triangle* Δ *but at the same moment passing through all of its neighboring triangles (the three triangles* $\Delta_a, \Delta_b, \Delta_c$ *sharing the sides with* Δ*).*

Proof. Indeed, if such a trajectory exists the vertices A, B, C of Δ are colored in the same color and the other three vertices $A', B', C'$of $\Delta_a, \Delta_b, \Delta_c$ not belonging to Δ are colored in an opposite color. By looking at the system in a circumcircle, one remarks that the image of Δ in the circle lies on one side of the chord corresponding to the trajectory. In this case, its reflection with respect to at least one of its sides also lies on the same side of this chord. This reflection is an image of one of the triangles $\Delta_a, \Delta_b, \Delta_c$. This forces one of the vertices A', B', C' to have the same color as the vertices of Δ. We obtain a contradiction. □

The proof of this Proposition is simple but we failed to generalize it in order to prove Conjectures 1 and 2. The difficulty consists in the fact that the chord in the circle doesn't correspond to one trajectory but to a family of such trajectories. This conjecture can be reformulated in the language of topology: in one folds all of the triangles in the triangle tiling on to a circle and then cuts along

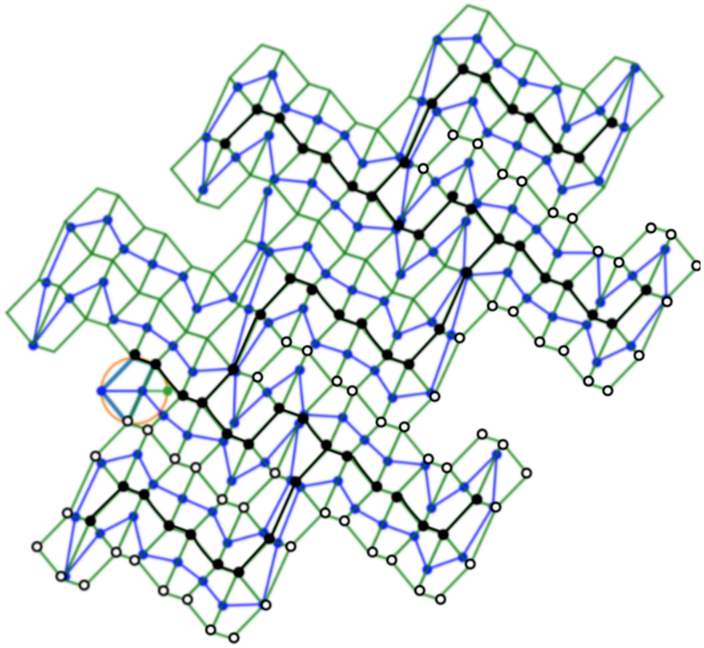

FIGURE 5. The generalisation of the Tree Conjecture for cyclic quadrilaterals seems to also hold experimentally. This Figure represents a part of the trajectory (in blue) in a billiard with a graph $\Gamma_{\mathcal{B}}$ drawn completely for this part of trajectory and the graph $\Gamma_{\mathcal{W}}$ drawn in part (to be able to show the progression of the growth of the graphs).

a chord in this circle, the plane will split in some number of surfaces, and none of these surfaces is annulus.

The tree conjecture still seems to hold experimentaly for the case of cyclic quadrilateral tilings, see Figure 5. The density conjecture also holds modulo the following minor changes. Now, if trajectory comes from one edge to another, it may be possible that the two vertices of the same color can not be connected by the edge in a tiling (in the case when the trajectory crosses two edges having a common vertex, for example). In this case one includes also the last vertex and one connects this vertex to its neighbours and one colors them all in the same color. In this way, all the crossed quadrilaterals will also have all of their vertices colored.

Note that tree conjecture is false in full generality on locally foldable tilings, as shows this simple remark.

Proposition 3. *The triangle tiling is locally foldable and can be modified locally into a new locally foldable tiling on which tiling billiard trajectories can contour the tiles.*

Proof. Consider a standard triangle tiling (even with an equilateral triangle), and a 6-periodic trajectory in it. Define a new tiling which is the obtained by cutting the plane by three families of equidistant parallel lines plus six segments that are the parts of the simple 6-periodic trajectory. By definition, a tiling obtained in such a way is locally foldable. One can see, that there is an obvious 6-periodic trajectory in such a tiling that doesn't verify tree conjecture, see Figure 6. Moreover, there is an open set of counterexamples (if one changes all of the parameters by a little, the combinatorics stays the same). □

One suspects that this is the only way how the tree conjecture may be false (existence of closed loops in the boundaries of the tiles that are folded inside a straight line).

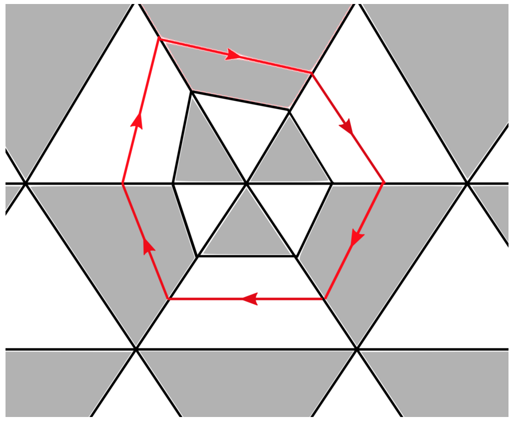

FIGURE 6. Simple counterexample to the tree conjecture for locally foldable tilings.

2.2. **Reducing the tree conjecture to the flower conjecture (but not proving flower conjecture...)** There are two natural families of foliations that are defined for tiling billiards on locally foldable tilings that come from the line foliations on the folded construction.

Definition 1 (Foliation $\mathcal{F}_1$)**.** Suppose that a locally foldable tiling is folded in some origami. Take a sheaf of oriented parallel lines on the plane. Then one can consider how they intersect the polygons in the origami and hence, initial polygones on the plane.

Definition 2 (Foliation $\mathcal{F}_2$)**.** Suppose that a locally foldable tiling is folded in some origami. Take a sheaf of lrays going out (propagating) from some fixed point (that we call **base point**) inside one of the tiles. Then one can consider how they intersect the polygons in the origami and hence, initial polygones on the plane.

Question 2. Study the *Illumination problem* for tiling billiards. For a fixed tiling, does there exist a finite number of points such that the light coming out from these points and following the tiling billiard negative refraction law, illuminates all the plane ? If yes, how many points of that kind are needed ?

Remark. Even for a square tiling, the anwer to this problem doesn't seem obvious. For a point being a center of a square τ_0, one can see that it illuminates a regular subset of a column and a row containing τ_0. For a point on one of the sides, the illuminated set is also quiet obvious. Although, for a point chosen arbitrary, the illuminated set seems to be more complicated. This can be interesting to study by using computer simulations.

By Lemmas 1 and 2, one can easily see that these foliations are well defined.

Lemma 3. a. *Foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ being parallel foliation and from-one-point foliation, are well defined.*

b. *Each of the leaves of the foliation is a trajectory of a triangle tiling billiard (possibly singular).*

c. *Consider some foliation of type $\mathcal{F}_1$ and two of its leaves (trajectories, possibly singular, of a tiling billiard). Then if these two leaves enter the same vertex, they correspond to the same line on a folded construction.*

Proof. The points a. and b. follow from Lemmas 2 and 1, and[c.] is obvious since it's exactly the leaf passing through this vertex in the folded system. ☐

Remark. Note that any triangle tiling trajectory can be included in some foliation of type $\mathcal{F}_1$ (this foliation is defined uniquely by a chosen trajectory) and in a family of foliations $\mathcal{F}_2$ (by choosing any point on the trajectory as the base point).

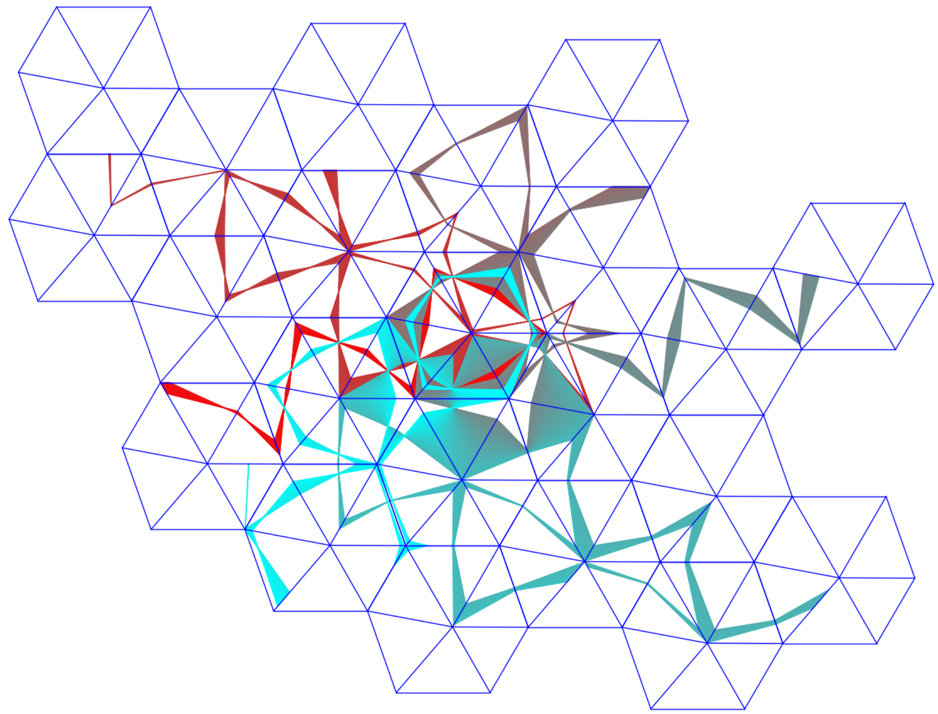

FIGURE 7. *Some parts of the leaves of the foliation* $\mathcal{F}_2$ starting in a fixed point in some triangle. Different colours correspond to different futures. Note that the foliation $\mathcal{F}_2$ as it is defined in this note is not the same as a lamination which is defined as union of all light beams going from one point on the plane in all directions. Indeed, the second lamination doesn't foliate all the plane but only a part of it, leaving out some zones that won't be *illuminated*. This remark touches on the questions of *Illumination problem* in the context of tiling billiards that has not been yet considered by anyone. We thank Julien Lavauzelle for this picture.

Both of the classes of foliations $\mathcal{F}_i, i = 1, 2$ consist of oriented foliations with singularities in all the vertices of the system. Note that these foliations can be restricted to the invariant subset of the billiard map, e.g. the interior of any closed trajectory is foliated by the foliations from these classes if one takes such parameters such that the closed trajectory is a leaf of these foliations.

Example. It is interesting to see the foliation $\mathcal{F}_2$ with a base point in the vertex of one of the added tiles (not the middle vertex). One can see that the loop going around the vertex in the midle made by the boundaries of tiles can be considered as a degenerate leaf of the foliation $\mathcal{F}_2$ that can't be oriented. The existence of such leaves is an obtruction for the Tree conjecture, see Figure 8

One can see that on this example, there exist two separatrix loops in one vertex belonging to the same oriented foliation $\mathcal{F}_1$ such that one of them contains the other. We conjecture that for the triangle tiling billiards this is not possible and prove that this is the sufficient condition in order for the Tree conjecture to hold.

In other words, we reduce the Tree conjecture to the following

Conjecture 3 (Flower conjecture)**.** *For a triangle tiling billiard, two separatrix loops in one vertex, they have the same orientation as curves with respect to infinity. The open domains that they bound are disjoint.*

Let us give a remark on the name. One can see that for any fixed foliation $\mathcal{F}_1$, each vertex of this foliation is a singular point. The number of separatrices of such a point is even (since the foliation is

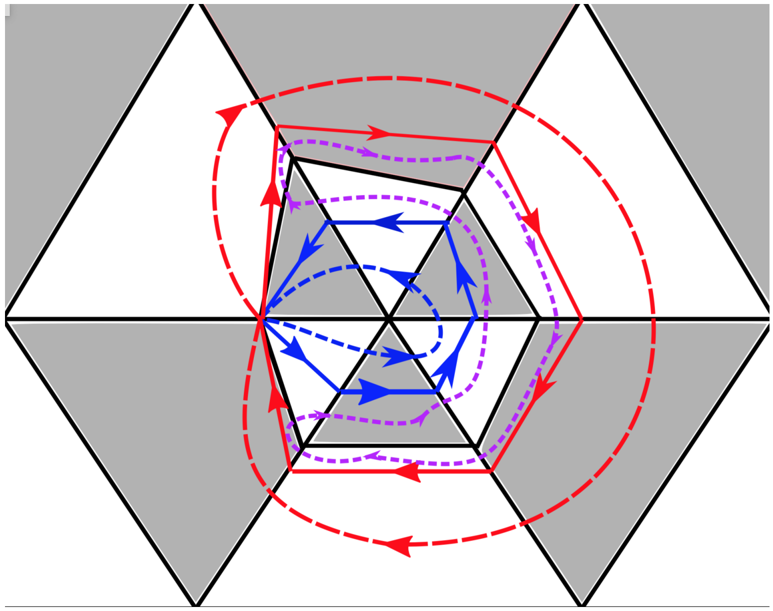

FIGURE 8. Behavior of the trajectories of the foliation $\mathcal{F}_2$ for the tiling defined as a small perturbation of triangle tiling, see Proposition 3. In such a new locally foldable tiling one can see that the local behavior of the foliation $\mathcal{F}_2$ centered at one of the singular points is the following. Inside the inserted six tiles the trajectories move counter-clockwise. Outside the tiles and close to the tiles the trajectories in $\mathcal{F}_2$ move clock-wise. One can choose one blue and one red trajectory in such a way that they belong to the same foliation $\mathcal{F}_1$. Then, the area between two these trajectories can be foliated by the closed trajectories of period 12.

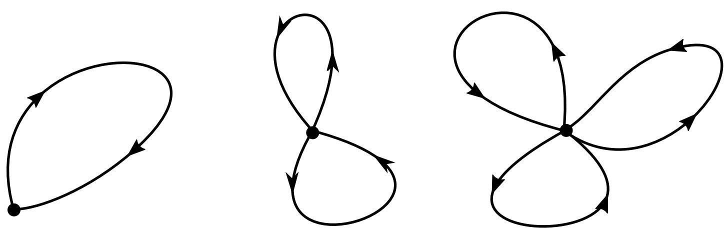

FIGURE 9. Possible local behavior of separatrices of triangle tiling billiards, in the case if the Flower Conjecture holds.

oriented), in other words, $0, 2, 4$ or 6. Suppoe that a vertex is contained inside some closed trajectory. Hence, in a corresponding foliation $\mathcal{F}_1$ all the separatrices of this vertex have to eventually come back to themselves and form closed loops. If the Flower conjecture holds, one of the three local pictures of the behavior of the separatrices is possibl, see Figure 9.

Theorem 1. *If the Conjecture 3 holds then the Conjecture 1 holds.*

Proof. For a periodic trajectory, it is contained in the annulus of parallel periodic trajectories in $\mathcal{F}_1$. One contracts it inside the zone bounded by it to obtain a trajectory that bounds a smaller volume, tille the moment when this leaf is a separatrix or the union of the separatrices. One looks at the boundary of the maximal annulus. A separatrix entering the same vertex from which it goes out (separatrix loop) is a well defined trajectory - a singular trajectory. Then, a local behavior of separatrices in one vertex can be one of four kinds - see Figure 9 or a point. If we reduced the behavior to the point, the initial trajectory has period 6 and satisfies tree conjecture. If not, one can proceed by induction. Since the synbolic dynamics of the periodic trajectory is defined by the symbolic dynamics

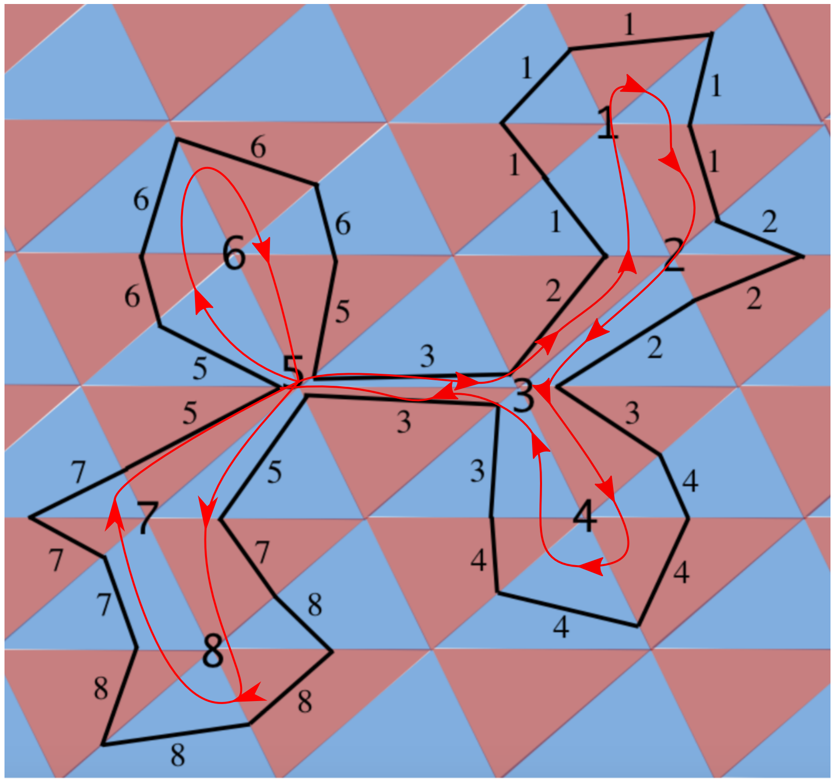

Figure 10. A Figure from [9] with a first step of induction for Flower Conjecture drawn.

of the separatrices inside it and since topologically, one of the four pictures is possible on each of the steps (0, 1,2 or 3 petals). Each vertex is singular hence by each vertex passes a separatrix of $\mathcal{F}_1$. For each periodic trajectory inside, the number of vertices inside it diminishes at least by one. Each trajectory contains at least 1 singular point. If n vertices, degenerate and take the union. Then the symbolic behavior $\omega = w_0 w_1 w_2$ where w_j are symbolic behaviors of periodic trajectories characterizing the petals, see Figure 10. □

Remark. A simpler proof for obtuse triangles (than in [9]) if flower conjecture holds is given by the fact that each flower for obtuse triangle tiling has at maximum two petals.

If Flower conjecture holds for some part of the plane, there one has tree conjecture.Tree conjecture is completely true if flower conjecture is. For ANY locally foldable tiling.

In general, one could understand the geometry of graphs inside if the tree conjecture doesn't hold (see read notebook no-dec 2018) Union of graphs.... This is based on ny idea of what makes conjecture flower not work but I maybe wrong since my proof of it is wrong...

3. Invariants of Rauzy graphs for interval exchange transformations with flips

As shown in [9, 25], the dynamics of triangle tiling billiards can be reduced to the one-dimensional dynamics of interval exchange transformations with flips on the circle. In the way analogous to that of the standard Rauzy induction for IET, the modified Rauzy induction can be defined for interval exchange transformations with flips [37]. Although, a geometric and combinatorial study of Rauzy graphs corresponding to this modified Rauzy induction has never been conducted. The reason being that for an IET with flips, one may not necessarily construc an associated flat surface through Rauzy-Veech zippered rectangles construction - in some cases, these surfaces simply just do not exist. Although, as we show with P. Hubert and P. Mercat, in the case of fully flipped IETs, these surfaces can indeed be constructed and seem to have beautiful combinatorial properties that we do not yet fully understand.

Using combinatorics of Rauzy classes, Kontsevich and Zorich classified the connected components of strata of the moduli spaces of Abelian differentials [30], and the case of linear involutions (and corresponding quadratic differentials) has been studied by Boissy-Lanneau [10]. These two studies use the natural geometric structures related to corresponding (generalized) interval exchange transformations: oriented flat surfaces with an oriented foliation in the case of Abelian differentials, and oriented flat surfaces with a non-oriented vertical foliation in the case of quadratic differentials. Inthe case of IETs with flips that is interesting to us, associated surfaces are *non-orientable*and have an oriented foliation which is constructed as a suspension of an IET with flips. This complexifies the study.

Problem 1. Generalize Rauzy-Veech zippered rectangles construction of a flat surface associated to an IET to the case of an IET with flips. Classifty those IETs with flips for which such a surface exists.

Problem 2. Find combinatorial invariants of Rauzy graphs of (certain) IETs with flips, and obtain the classification of connected components in the corresponding moduli space.

We are now intensively working on these two problems with P. Mercat and P. Hubert, so I do not go into details here in hope that we finish the study, and write it up in a nice way. These two problems are connected to the following one that was studied, among others, in [44].

Problem 3. Caracterise those elements of $\mathrm{IETF}^n[0,1]$ which are minimal.

We hope to answer this question, at least in the case of 5 intervals: make the combinatorial structure as well as the lengths of such intervals explicit. For the case of 3 intervals (or less), none of the elements of $\mathrm{IETF}^n[0,1]$ is minimal: the modified Rauzy induction always stops, and there is always a periodic cylinder. For the case of4 intervals, the question is solved in [25] and we prove that the minimal transformations correspond to the combinatorics coming from 3-interval fully flipped transformation on the circle coming from tiling billiards. Moreover, the lengths of the intervals are parametrized by the Rauzy gasket. For $n = 5$, we hope to obtain a similar answer in terms of cyclic quadrilateral tilings and obtain a definition of the Rauzy gasket in higher dimension as well as find invariant measures for this set, following [7, 8]. Following some simulations we already did, we think that starting from 6 intervals, new phenomena can appear, and the thermodynamic formalism will be much more complicated.

4. Negative refraction in the wind-tree model

In a current project with D.Davis and F. Valdez, we decided to adapt a famous wind-tree model in the context of tiling billiards with refractions. This model was proposed by Paul and Tatiana Ehrenfest in the context of statistical mechanics, and since then has been studued a lot from the mathematical point of view [24, 14].

We consider a following adaptation of this model : we study a billiard in the euclidian plane with obstancles in the form of rectangles placed in the points of the lattice Λ of the plan. We know suppose that a ball can enter *inside* the obstacles and then get out of them, and in doing so, the trajectory will follow the refraction coefficient equal to -1. We are interested in the dynamics of this billiard.

Note that there exists a natural non-oriented foliation defined with respect to these trajectories.

A lattice is (a, b)-**admissible** if the obstacles of size $a \times b$ put in the points of such a lattice do not intersect. A direction $\theta \in \mathbb{S}^1$ is **trapped** if any trajectory parallel to this direction on the exterior of the obstacles is contained in some band in the plane. We hope to prove the following conjecture, see Figure 3 for illustration of one trajectory.

Conjecture 4. *For all a, b and all lattices Λ that are (a, b)-admissible, almost any direction $\theta \in \mathbb{S}^1$ is trapped.*

We also hope to construct the examples of lattices Λ and directions θ which are *ergodic* and do not conform to a general rule.

This dynamical system has a strong connection (in its dynamical behavior as well as in its geometric meaning and symmetries) to Eaton lenses system, for which the analogue of Conjecture 4 has been proven in [17, 18, 19]. The arguments of the proofs are the following : first, the authors construct an infinite translation surface $\tilde{M}$ on which the dynamics of a vertical flow corresponds to a studied billiard. Then, by using symmetries, one can pass to the double cover and obtain a compact translation surface M. The last (and most technical) step is to establish the dictionnary between moduli spaces of double covers and the space of affine lattices. This permits Birkhoff and Oseledets genericity and, finally, to apply the machinerie of Kontsevich-Zorich cocycles to this problem. We hope to apply a similar strategy to the wind-tree case. The difficulty is that e $M \in \mathcal{H}(1,1)$ for Eaton lenses and $M \in \mathcal{H}(2,2)$ in our case. We are in a harder component and even more, the change of parameter θ for Eaton lense just gives a circle in moduli space. In our case, we obtain a more difficult curve. We still hope that genericity criteria established by Fraczek-Shi-Ulcigrai can be applied, by using the work by Avila-Eskin-Möller[6] and Eskin-Filip-Wright[16].

5. From the point of view of geometric group theory: SAF invariant and fully flipped IET

One can interest ourselves in the dynamics of IETs with flips from the point of view of geometric group theory. A couple of elements from IETF^n - what group do they form ?

I would like to thank Yves Cornulier for the following valuable remark.

Lemma 4. *Let X be the set of all maps in* IET *such that there exists a fully flipped interval exchange transformation F for which $X = F^2$. Then the subgroup of* IET *generated by X coincides with the kernel* $\text{IET}_0 := \{F \in \text{IET} : SAF(F) = 0\}$ *of the SAF invariant.*

Proof. $\text{IET} \subset \text{IET} \rtimes \mathbb{Z}/2\mathbb{Z} = \text{IET} \sqcup \text{FET}$. As proven in [25], Proposition 18, $X \subset \text{IET}_0$. The set X is invariant by conjugation by elements in $\text{IET} \sqcup \text{FET}$. Hence X is invariant by conjugation in IET_0 (as its subgroup). Hence $< X >$ us a normal subgroup of IET_0. But IET_0 is simple hence $< X >= \text{IET}_0$. □

This completes a description of IET_0 given in [46] by Vorobets.

Definition 3. A group G is simple if $\forall g \in G, g \neq 1$ and $\forall h \in G$ there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in G$, $e_1, \ldots, e_n \in \{\pm 1\}$ such that

$$h = x_1 g^{e_1} x_1^{-1} \ldots x_n g^{e_n} x_n^{-1}. \tag{1}$$

A group G is uniformly simple if there exists a uniform $n \in \mathbb{N}$ such that $\forall g \in G, g \neq 1$ and $\forall h \in G$ there exist $x_1, \ldots, x_n \in G$, $e_1, \ldots, e_n \in \{\pm 1\}$ such that (1) holds.

Question 3. Is the group IET_0 uniformly simple ?

Question 4. There is a natural family of fully flipped IETs related to triangle tiling billiards called CET^3_τ. For $\tau = \frac{1}{2}$ the involutions to take in order to write a map as a product of involutions are obvious. But in general case ?... For any τ, and how many are they ?

References

[1] P. Arnoux, J. Bernat, X. Bressaud, *Geometrical models for substitutions*, Exp. Math. 20, 97–127 (2011)

[2] P. Arnoux, *Un exemple de semi-conjugaison entre un échange d'intervalles et une translation sur le tore*, Bull. Soc. Math. France 116, 489–500 (1988)

[3] P. Arnoux, *Un invariant pour les échanges d'intervalles et les flots sur les surfaces*, doctoral thesis (1981)

[4] P.Arnoux, G. Rauzy, *Représentation géométrique des suites de complexité* $2n + 1$, Bulletin de la SMF., 119: 2, 199–215 (1991)

[5] P.Arnoux, S. Starosta, *The Rauzy gasket*, Birkhäuser Boston. Further Developments in Fractals and Related Fields, Springer Science+Business Media New York, 1–23, Trends in Mathematics (2013)
[6] A. Avila, A. Eskin, M. Moeller *Symplectic and Isometric* $\mathrm{SL}(2,\mathbb{R})$ *invariant subbundles of the Hodge bundle*, Crelle's Journal, 732 (2017)
[7] A. Avila, P. Hubert, A. Skripchenko, *Diffusion for chaotic plane sections of* 3*-periodic surfaces*, Inventiones mathematicae,volume 206, issue 1, 109–146 (2016)
[8] A.Avila, P. Hubert, A. Skripchenko, *On the Hausdorff dimension of the Rauzy gasket*, Bulletin de la société mathématique de France, 144 (3), pp.539 - 568 (2016)
[9] P.Baird-Smith, D.Davis, E.Fromm, S.Iyer, *Tiling billiards on triangle tilings, and interval exchange transformations*, preprint, `http://www.swarthmore.edu/NatSci/ddavis3/triangle_tiling_billiards.pdf` (2018)
[10] C.Boissy, E. Lanneau *Dynamics and geometry of the Rauzy-Veech induction for quadratic differentials*, Ergodic Theory and Dynamical Systems (2008)
[11] Diana Davis, W. Patrick Hooper, *Periodicity and ergodicity in the trihexagonal tiling*, accepted pending revision in Commentarii Mathematici Helvetici (2018)
[12] D. Davis, K. DiPietro, J. Rustad, A. St Laurent, *Negative refraction and tiling billiards, to appear in Advances in Geometry* (2016)
[13] V. Delecroix, *Interval exchange transformations*, Lecture Notes, Salta (Argentina) (2016)
[14] Vincent Delecroix and Anton Zorich, *Cries and whispers in wind-tree forests* (2015), preprint
[15] R. De Leo, I. Dynnikov, *Geometry of plane sections of the infinite regular skew polyhedron* $\{4, 6 \mid 4\}$, Geom. Dedicata 138, 51–67 (2009)
[16] A. Eskin, S. Filip, A. Wright *The algebraic hull of the Kontsevich-Zorich cocycle*, preprint (2017)
[17] K. Fraczek, M. Schmoll, *Directional localization of light rays in a periodic array of retro-reflector lenses.* Nonlinearity 27 (2014), 1689–1707.
[18] K. Fraczek, M. Schmoll, *On ergodicity of foliations on* $\mathbb{Z}^d$*-covers of half-translation surfaces and some applications to periodic systems of Eaton lenses*, to appear in Communications in Mathematical Physics.
[19] K. Fraczek, R. Shi and C. Ulcigrai, *Genericity on curves and applications: pseudo-integrable billiards, Eaton lenses and gap distibutions.* J. Mod. Dyn. 12 (2018), 55–122.
[20] P. Glendinning, *Geometry of refractions and reflections through a biperiodic medium*, Siam J. Appl. Math., Society for Industrial and Applied Mathematics 76: 4, 1219–1238 (2016)
[21] *Je voudrais vous parler de mathématiques...*, short film co-created by C. Goudron and O. Paris-Romaskevich, for a competition Symbiose 48 hour film project, scientific documentary festival PariScience2018, https://https://vimeo.com/297265239 (2018)
[22] P. Hooper, Alexander St Laurent, *Negative Snell law tiling billiards trajectory simulations*, http://awstlaur.github.io/negsnel/
[23] W. Patrick Hooper, B. Weiss, *Rel leaves of the Arnoux-Yoccoz surfaces*, Selecta Mathematica, 24:2, 875-934 (2018)
[24] P. Hubert, S. Lelièvre, S. Troubetzkoy *The Ehrenfest wind-tree model: periodic directions, recurrence, diffusion*, J. Reine Angew. Math. 656 223–244 (2011)
[25] P. Hubert, O. Paris-Romaskevich *Triangle tiling billiards and the exceptional family of their escaping trajectories: circumcenters and Rauzy gasket*
[26] T. C. Hull *The combinatorics of flat folds: a survey*
[27] M. Keane, *Interval exchange transformations* Math. Z. 141, 25-31 (1975)
[28] R. Kenyon, W. Y. Lam, S. Ramassamy, M. Russkikh *Dimers and circle patterns* (2019)
[29] R. Kenyon, A. Okounkov, S. Sheffield*Dimers and Amoebae* (2003)
[30] M. Kontsevich, A. Zorich *Connected components of the moduli spaces of Abelian differentials with prescribed singularities*, Inventiones mathematicae, 153:3, 631–678 (2003)
[31] J. H. Lowenstein, G. Poggiaspalla, and F. Vivaldi, *Interval exchange transformations over algebraic number fields: the cubic Arnoux-Yoccoz model*, Dynamical Systems, 22(1), 73–106 (2007)
[32] S. Marmi, P. Moussa, J.-C. Yoccoz ,*Affine interval exchange maps with a wandering interval*, Proc. London Math. Soc. (3) 100, 639–669 (2010)
[33] Q. de Mourgues, *A combinatorial approach to Rauzy-type dynamics*, Université Paris 13, thesis (2017)
[34] A. Mascarenhas, B. Fluegel *Antisymmetry and the breakdown of Bloch's theorem for light*, unpublished draft
[35] C. McMullen, *Teichmüller geodesics of infinite complexity.* Acta Math. 191 (2003), no. 2, 191–223.
[36] C. McMullen, *Cascades in the dynamics of measured foliations*, Annales Scientifiques de l'École Normale Supérieure, 2012.
[37] A. Nogueira, *Almost all interval exchange transformations with flips are nonergodic*, Ergodic Theory Dynam. Systems 9:3, 515-525 (1989)

[38] S.P.Novikov, *The Hamiltonian formalism and multivalued analogue of Morse theory*, (Russian) Uspekhi Mat. Nauk 37: 5, 3–49 (1982); translated in Russian Math. Surveys 37:5, 1–56 (1982)
[39] M. Rao, *Exhaustive search of convex pentagons which tile the plane* (2017)
[40] G. Rauzy, *Échanges d'intervalles et transformations induites*, Acta Arith., 34(4):315–328, 1979.
[41] R. A. Shelby, D. R. Smith, S. Schultz, Experimental Verification of a Negative Index of Refraction, Science, Vol. 292 no. 5514, 77–79 (2001)
[42] B. Strenner, *Lifts of pseudo-Anosov homeomorphisms of nonorientable surfaces have vanishing SAF invariant*, Mathematical Research Letters, 25:2 (2018)
[43] D. Smith, J. Pendry, M. Wiltshire: Metamaterials and negative refractive index, Science, Vol. 305, 788–792 (2004)
[44] A. Skripchenko, S. Troubetzkoy, *On the Hausdorff dimension of minimal interval exchange transformations with flips*, Journal London Mathematical Society, to appear.
[45] J. Valentine, S. Zhang, T. Zentgraf, E. Ulin-Avila, D. A. Genov, G. Bartal and X. Zhang. Three-dimensional optical metamaterial with a negative refractive index. Nature, 455 (2008)
[46] Yaroslav Vorobets *Notes on the commutator group of the group of interval exchange transformations*